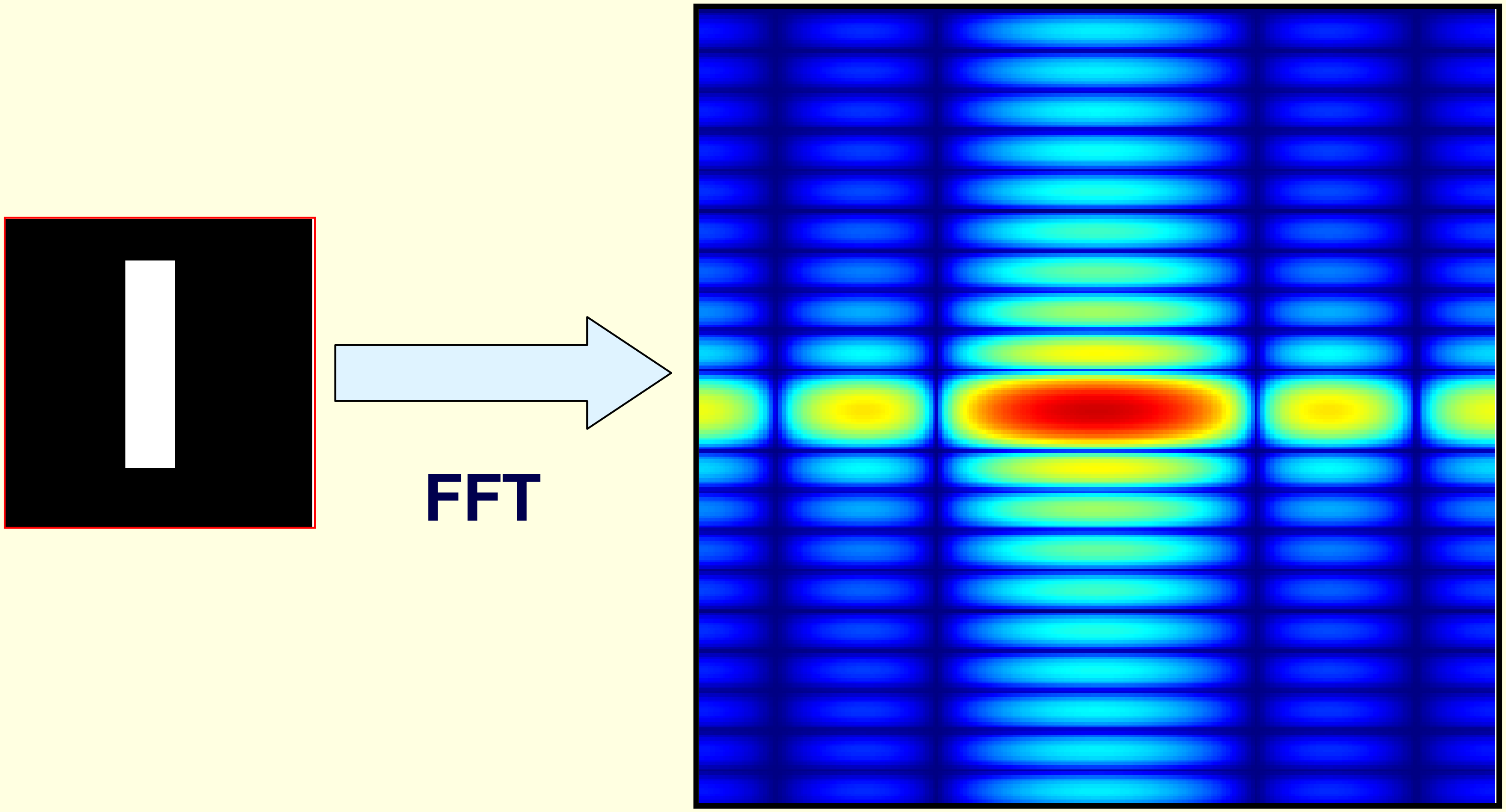
Fourier transform of images



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Fourier transform

Joseph Fourier has put forward an idea of representing signals by a series of harmonic functions

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

inverse

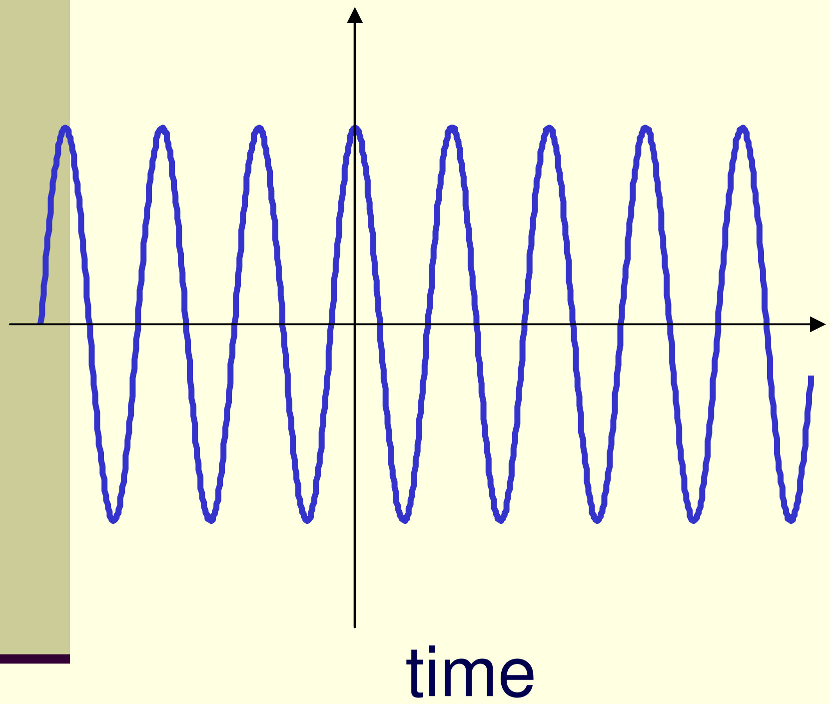
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

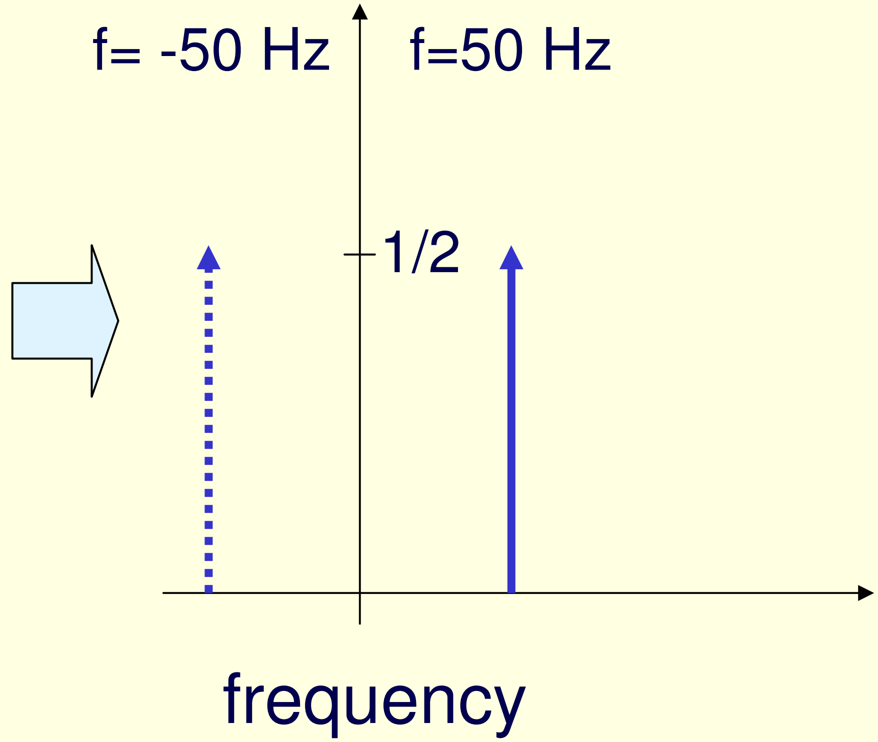
forward

Fourier coefficients

Joseph Fourier
(1768-1830)

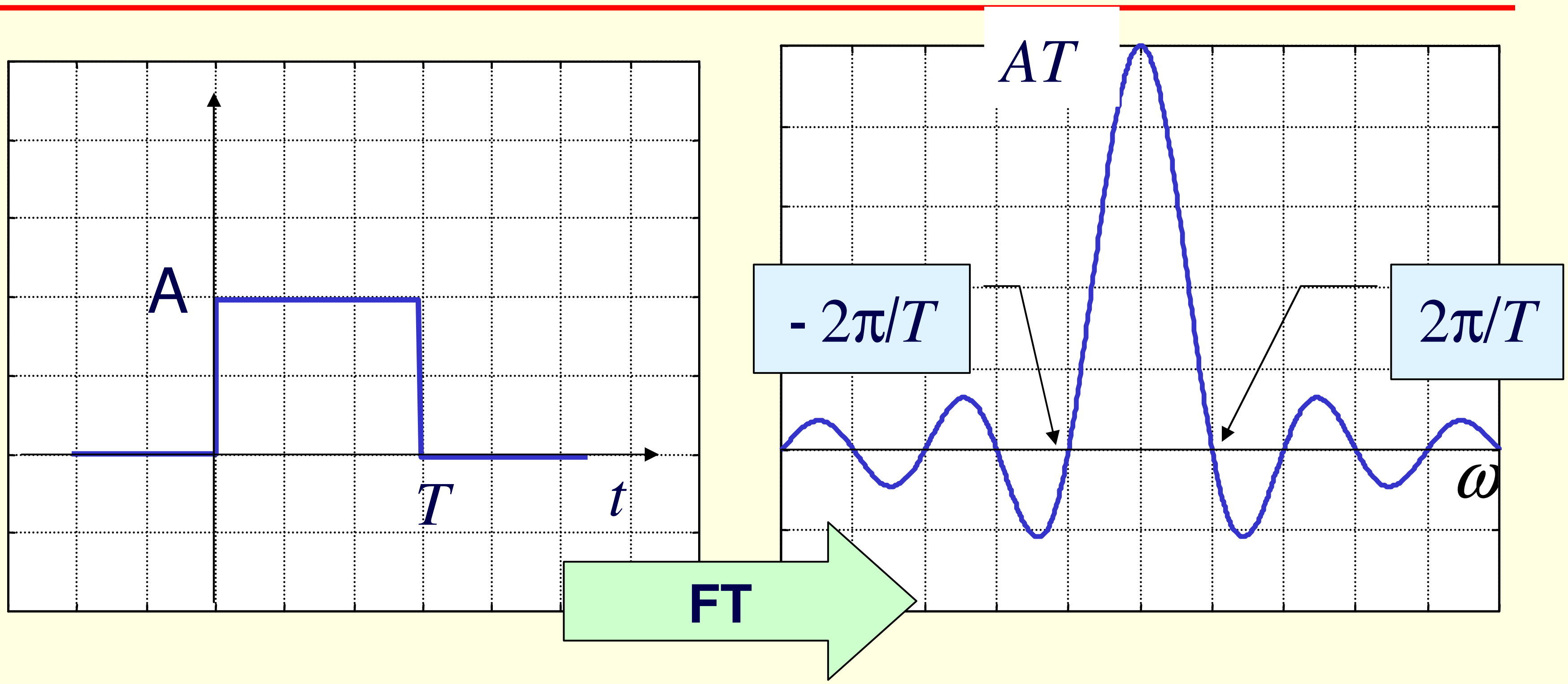
Fourier transform - example





$$X(j\omega) = \int_{-\infty}^{+\infty} \cos(\omega_0 t) e^{-j\omega t}\, dt = \frac{1}{2}\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

Fourier transform - example

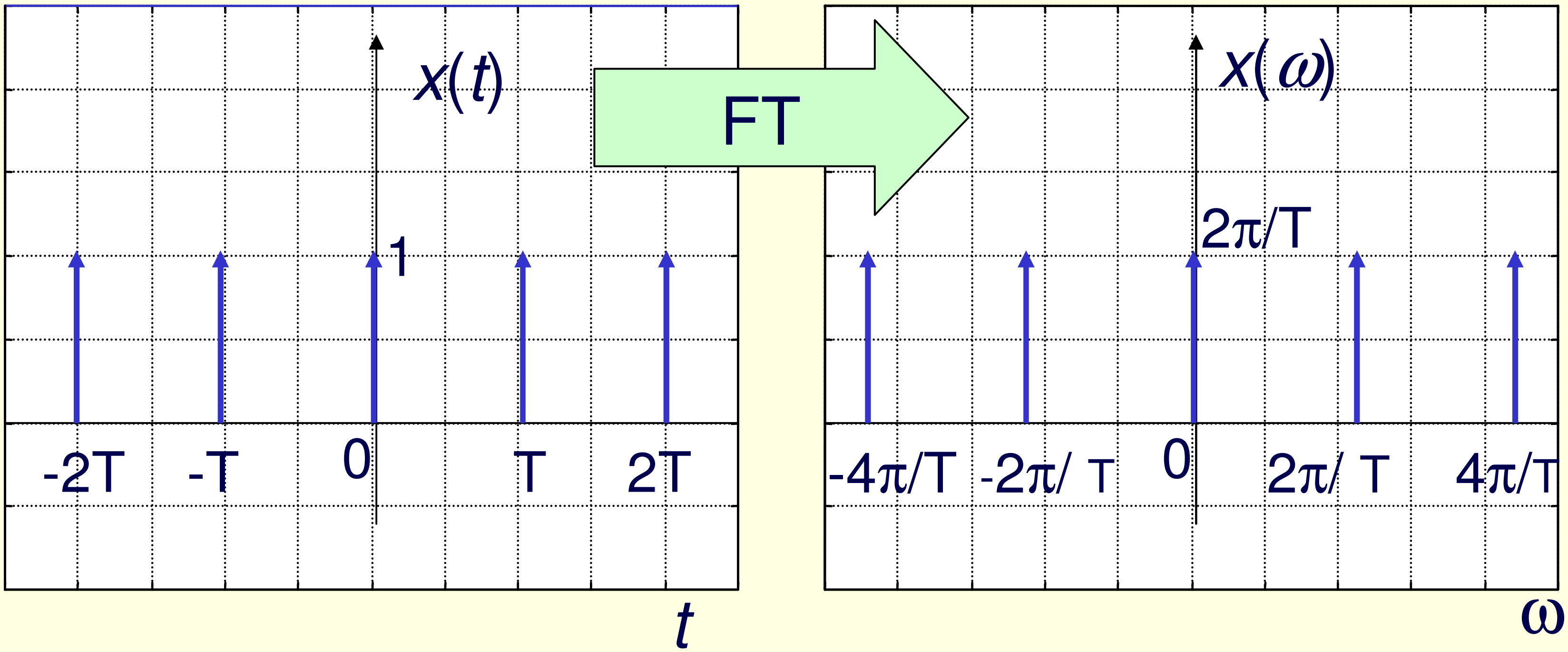


$$f(t)=\begin{cases}1, & t<T\\ 0, & t>T\end{cases}$$

$$F(\omega)=A\int_0^T e^{-j\omega t}\,dt=\frac{2A}{\omega}\sin\left(\frac{\omega T}{2}\right)e^{-j\frac{\omega T}{2}}$$

$$|F(\omega)|=?$$

Fourier transform - example



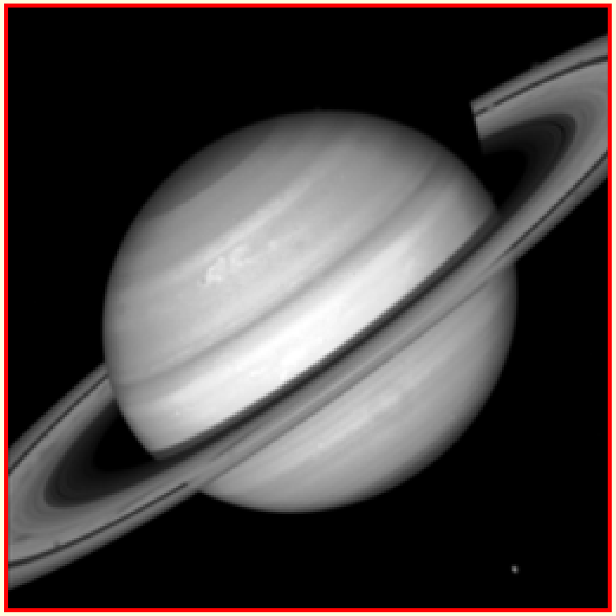
A series of Dirac pulses

$$\delta_T(t) = \sum_{-\infty}^{+\infty} \delta(t - kT) \quad \leftrightarrow \quad \omega_s \sum_{-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

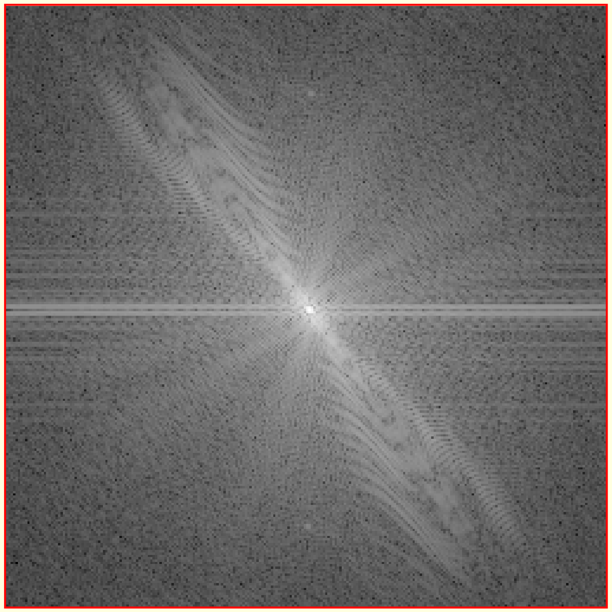
$$\omega_s = \frac{2\pi}{T}$$

Fourier transform of images

Monochrome image



Fourier spectrum



Why do we convert images (signals) to spectrum domain?

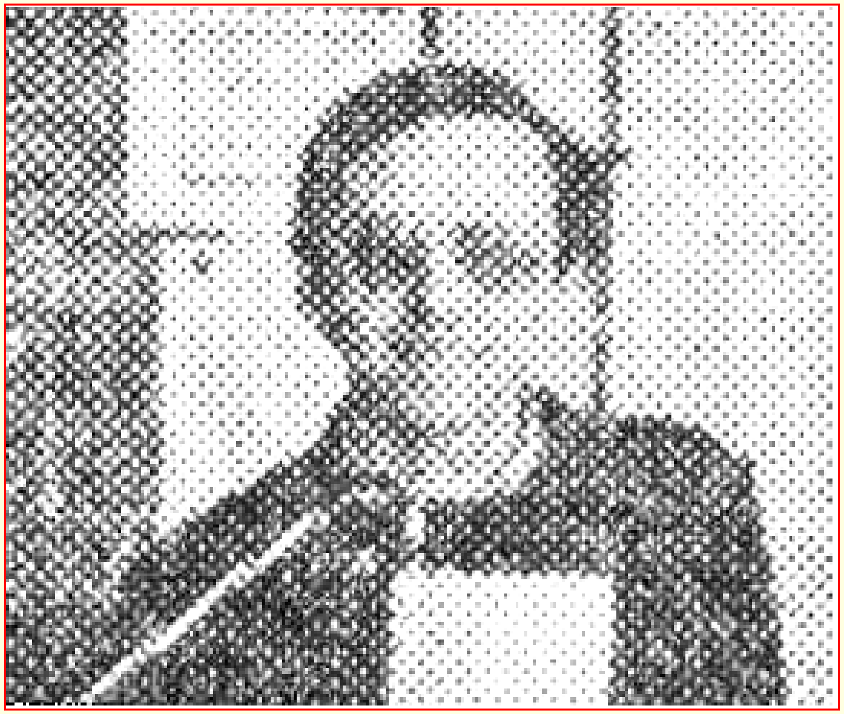
Fourier transform of images

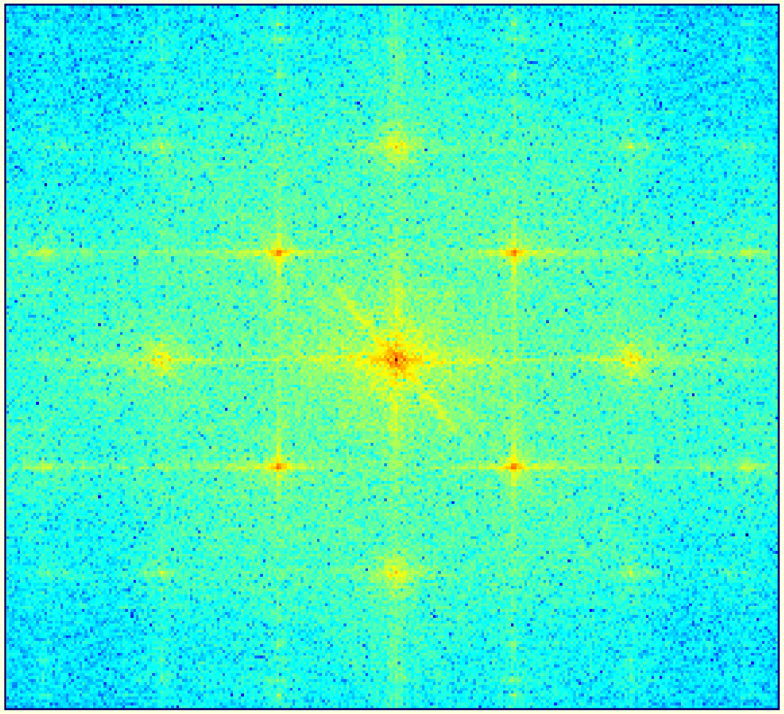
Why do we convert images to spectrum domain?

1. For **exposing image features** not visible in spatial domain, eg. periodic interferences
2. For achieving more compact image representation (coding), eg. **JPEG, JPEG2000**
3. For **designing digital filters**
4. For fast processing of images, eg. **digital filtering of images** in spectrum domain

Fourier transform of images

1. **Detection of image features**, eg. periodic interferences





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Fourier transform of images

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$ **forward**

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$ **inverse**

Euler equations?

$$\cos\omega_0 t=\frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)$$

$$\sin\omega_0 t=\frac{1}{2j}\left(e^{j\omega_0 t}-e^{-j\omega_0 t}\right)$$

Amplitude and phase spectrum of the Fourier transform of images

$$F(u,v) = |F(u,v)|\, e^{j\, arg[F(u,v)]}$$

$$|F(u,v)| = \sqrt{Re[F(u,v)]^2 + Im[F(u,v)]^2}$$

$$arg[F(u,v)] = arctan \frac{Im[F(u,v)]}{Re[F(u,v)]}$$

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The Discrete FT of images - I

$$F(u,v)=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)e^{-2\pi j\frac{ux}{N}}e^{-2\pi j\frac{vy}{N}}=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)e^{-2\pi j\frac{(ux+vy)}{N}}$$

$$dla \quad u,v=0,1,\ldots,N-1$$

Forward - FFT

$$f(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}F(u,v)e^{2\pi j\frac{ux}{N}}e^{2\pi j\frac{vy}{N}}=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}F(u,v)e^{2\pi j\frac{(ux+vy)}{N}}$$

$$dla \quad x,y=0,1,\ldots,N-1$$

Inverse - IFFT

$$e^{jk\omega_0 t}=e^{\frac{jk2\pi t}{T}}=e^{\frac{2\pi jkt}{T}}=e^{\frac{2\pi jk(\Delta tn)}{N\Delta t}}=e^{2\pi j\frac{kn}{N}}$$

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The Discrete FT of images - II

$$F(u,v)=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux+vy)/N}$$

$$dla \quad u,v=0,1,\ldots,N-1$$

$$f(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}F(u,v)e^{+j2\pi(ux+vy)/N}$$

$$dla \quad x,y=0,1,\ldots,N-1$$

Number of computations for 512x512 image?

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1D computational example

$$f(x) = [1\ 3\ 4\ 4]$$

$$N = 4$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$F(u) = \frac{1}{N}\sum_{x=0}^{x=3} f(x)e^{-j2\pi ux/N}$$

$$F(0) = \frac{1}{N}\sum_{x=0}^{N-1=3} f(x)e^{-j2\pi 0x/N} = \frac{1}{4}\left[f(0) + f(1) + f(2) + f(3)\right] =$$

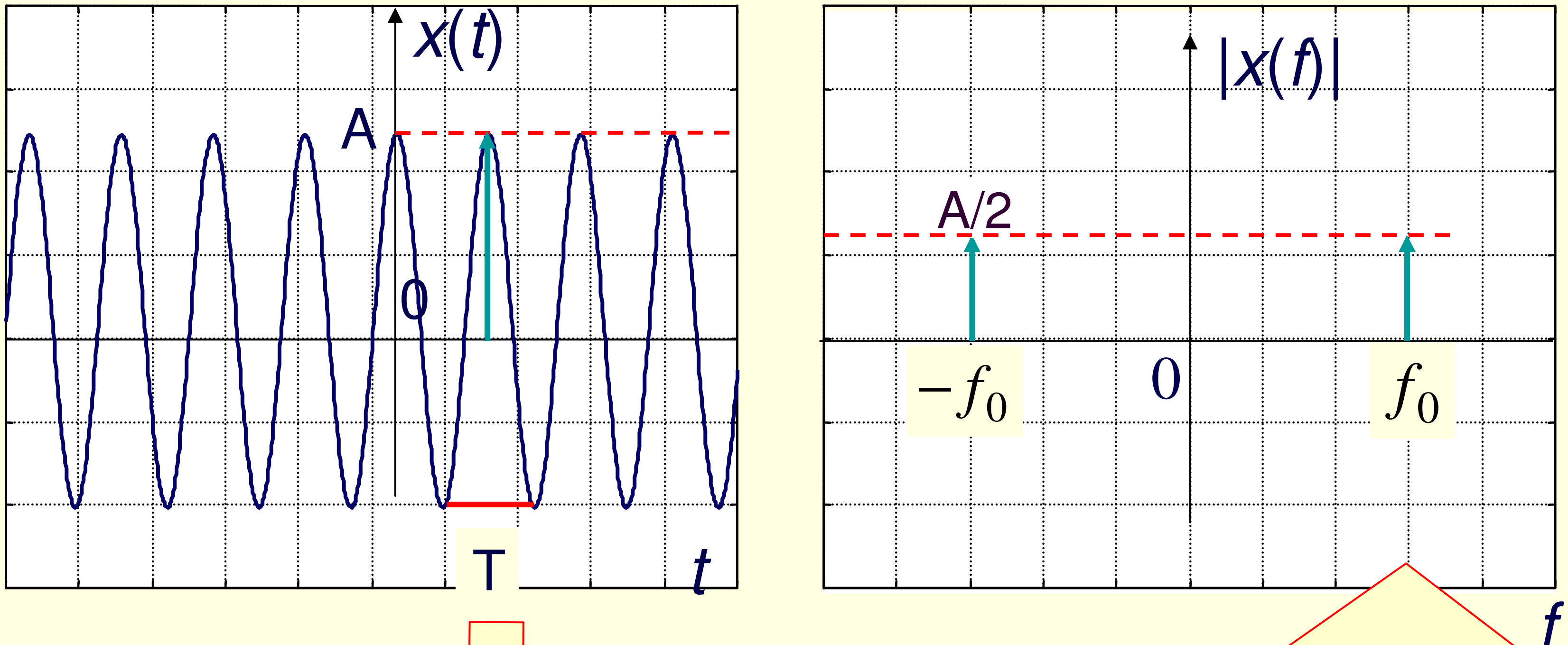
$$= \frac{1}{4}\left[1 + 3 + 4 + 4\right] = 3$$

$$F(1) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{4}(-3 + j)$$

$$F(2) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = -\frac{1}{4}(2)$$

$$F(3) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = -\frac{1}{4}(3 + j)$$

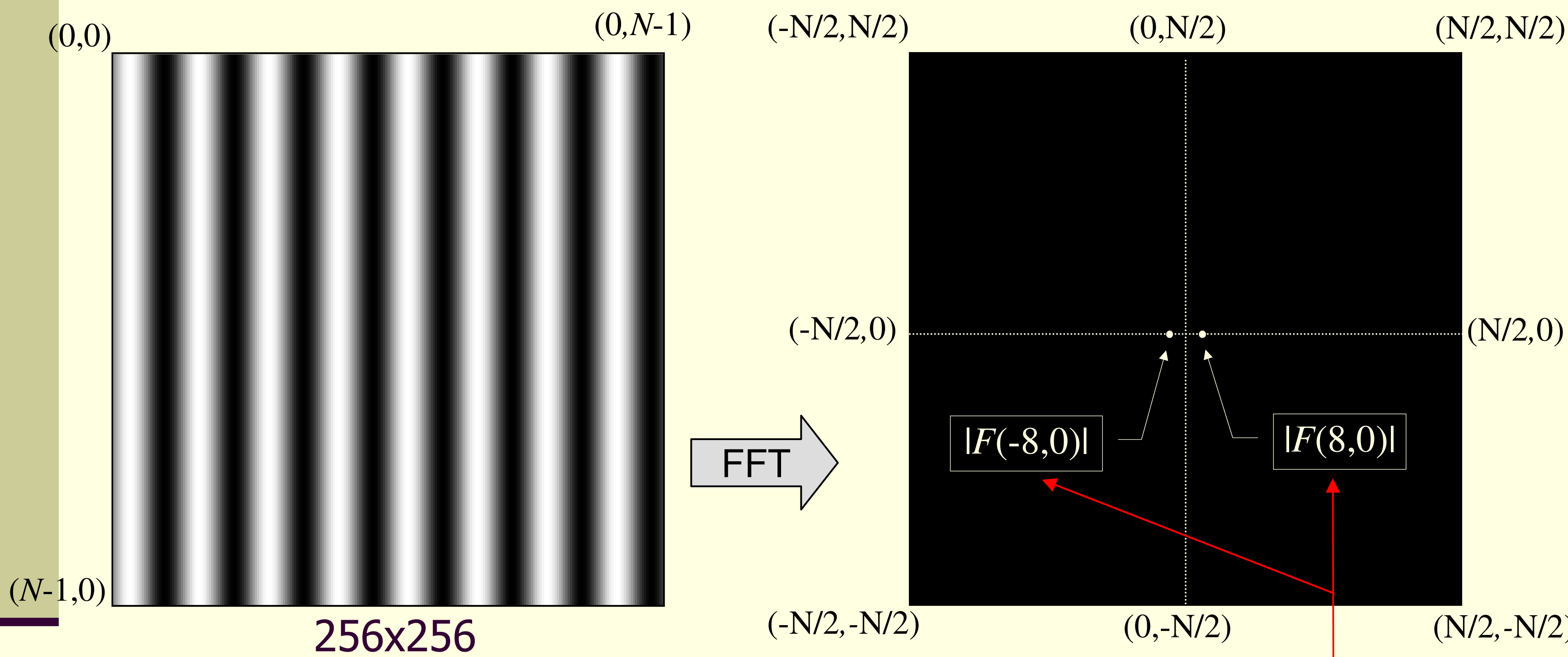
Cosine function Fourier spectrum



$$f_0 = \frac{1}{T}$$

time → **frequency**

Let us move to 2D



$T_{min\_period} = 2\Delta x$ is max possible frequency: $128\Delta u$

$T_{sin\_period} = 32\Delta x$ is 16x smaller frequency i.e.: $\frac{128\Delta u}{16} = 8\Delta u$

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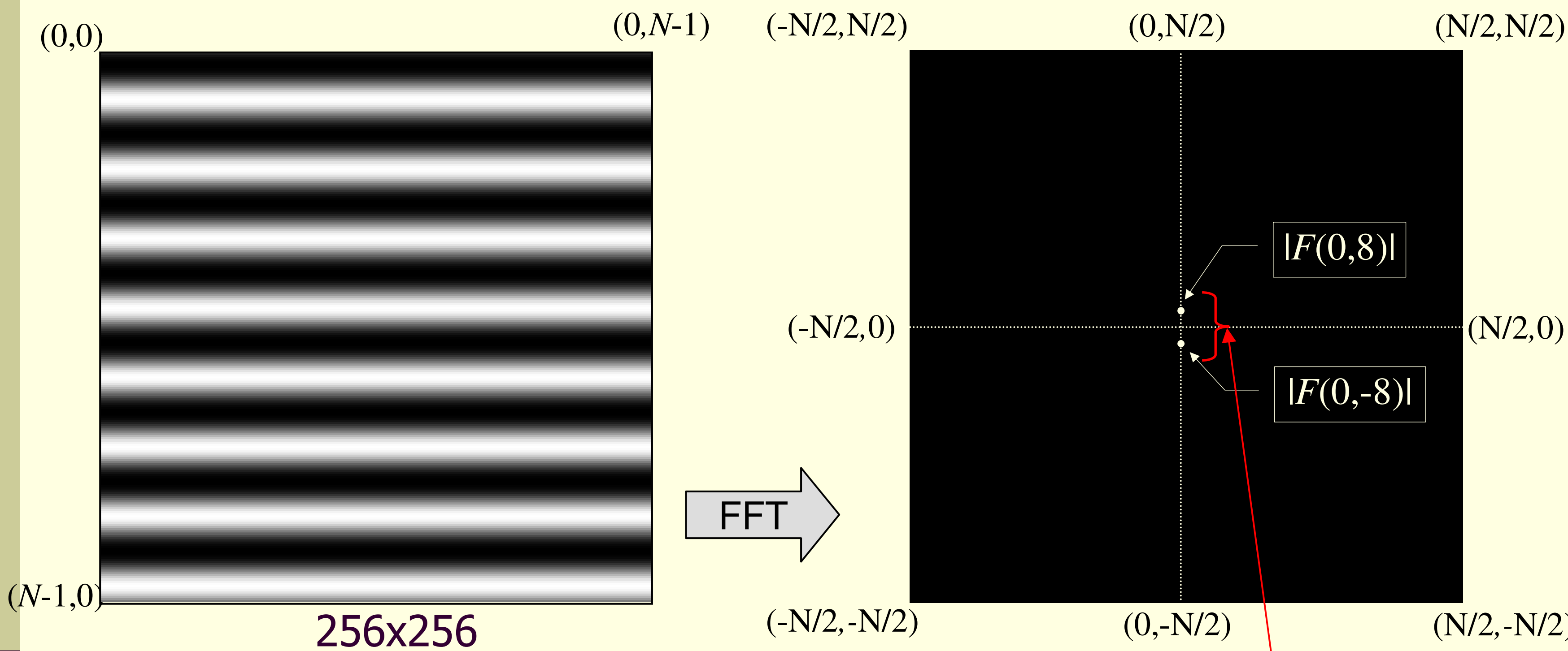
Let us move to 2D



$T_{min\_period} = 2\Delta x$ is max possible frequency: $128\Delta u$

$T_{sin\_period} = 32\Delta x$ is 16x smaller frequency i.e.: $\frac{128\Delta u}{16} = 8\Delta u$

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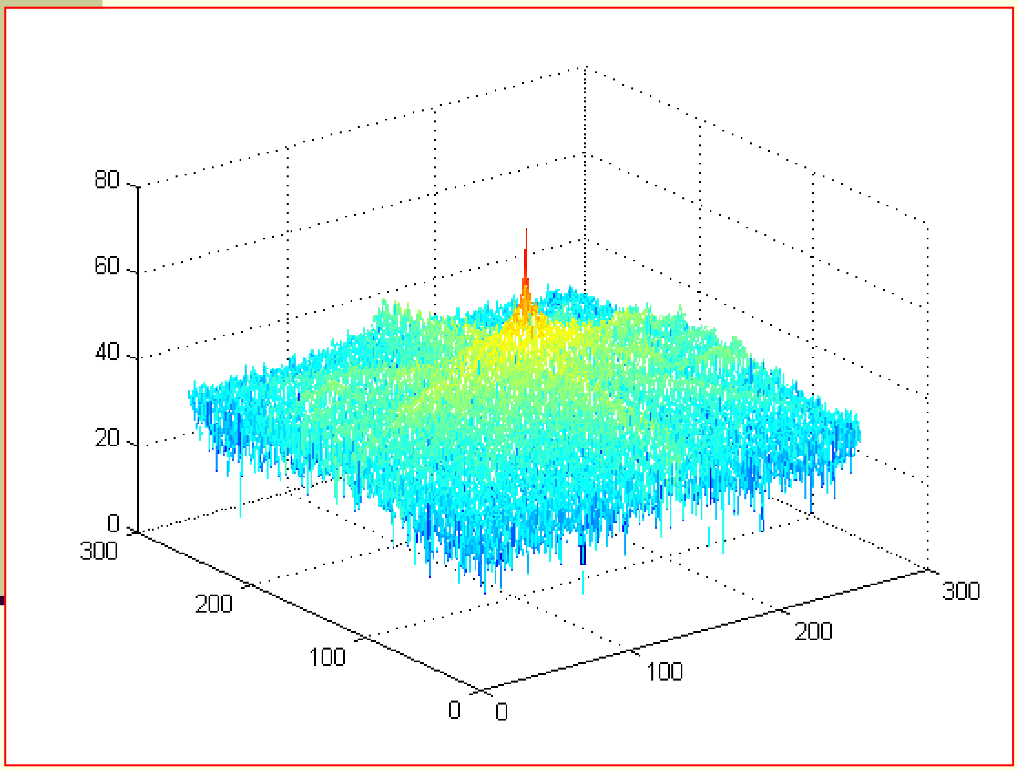
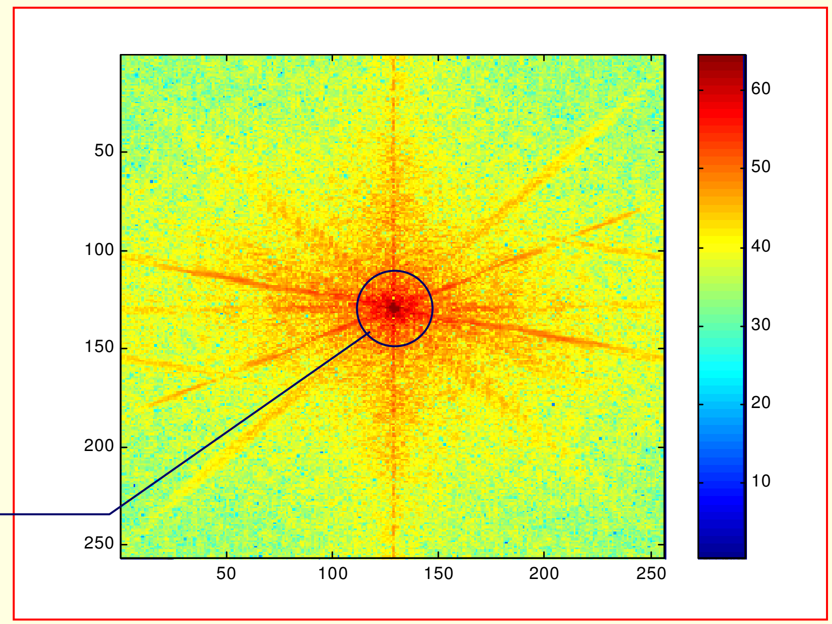
Fourier amplitude spectrum





~90% of spectrum energy

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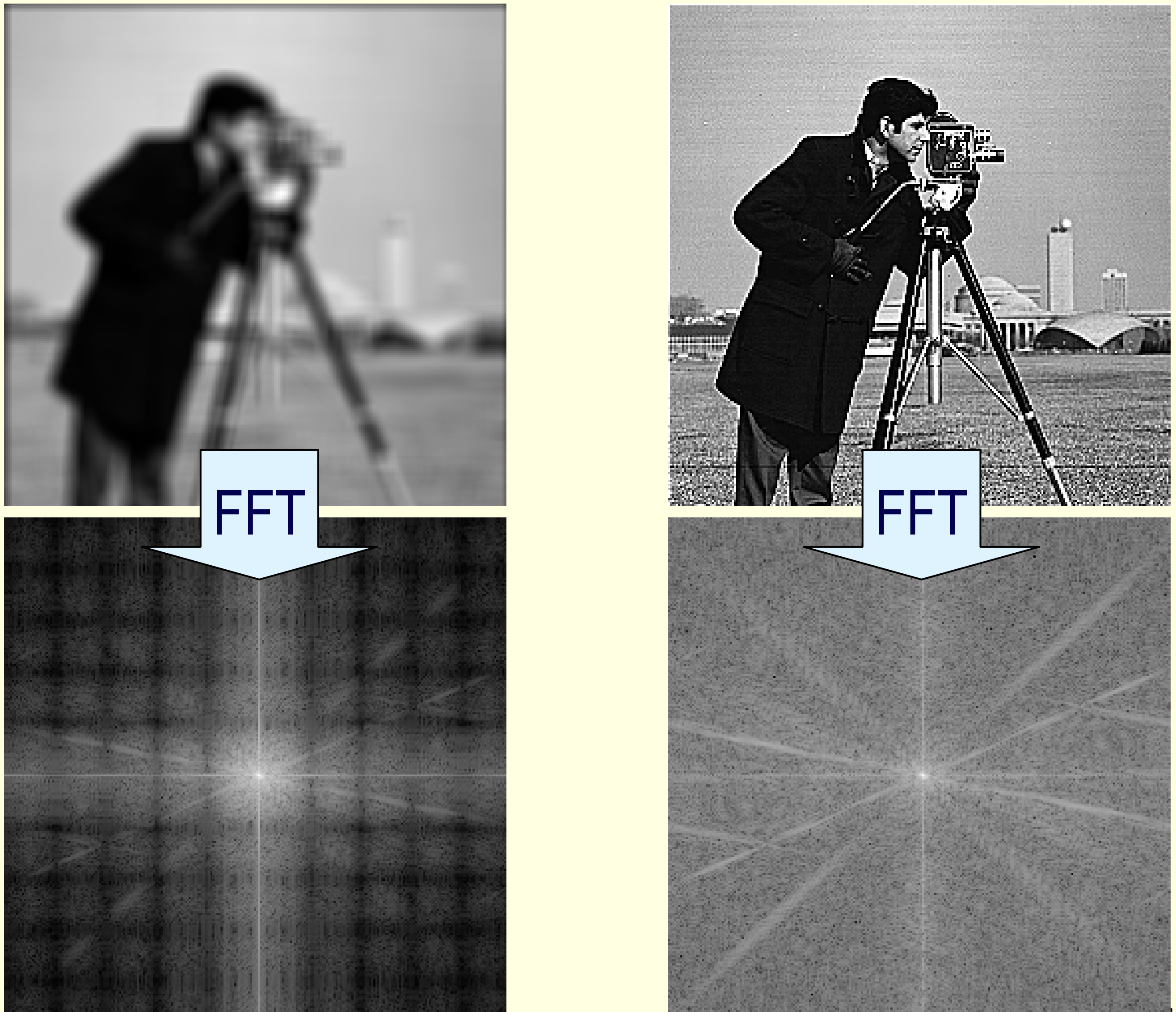
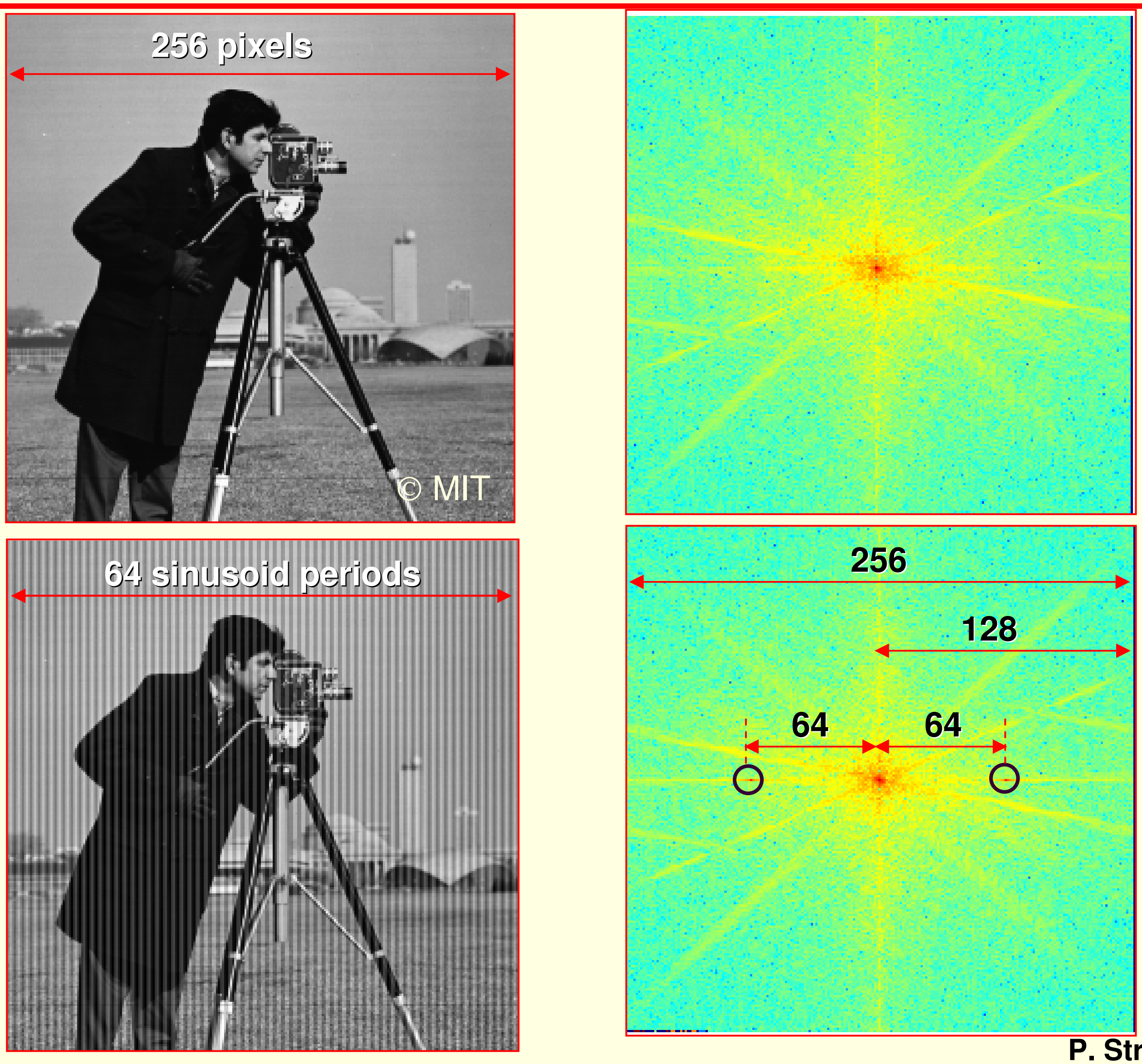
Fourier amplitude spectrum



Detection of periodic distortions



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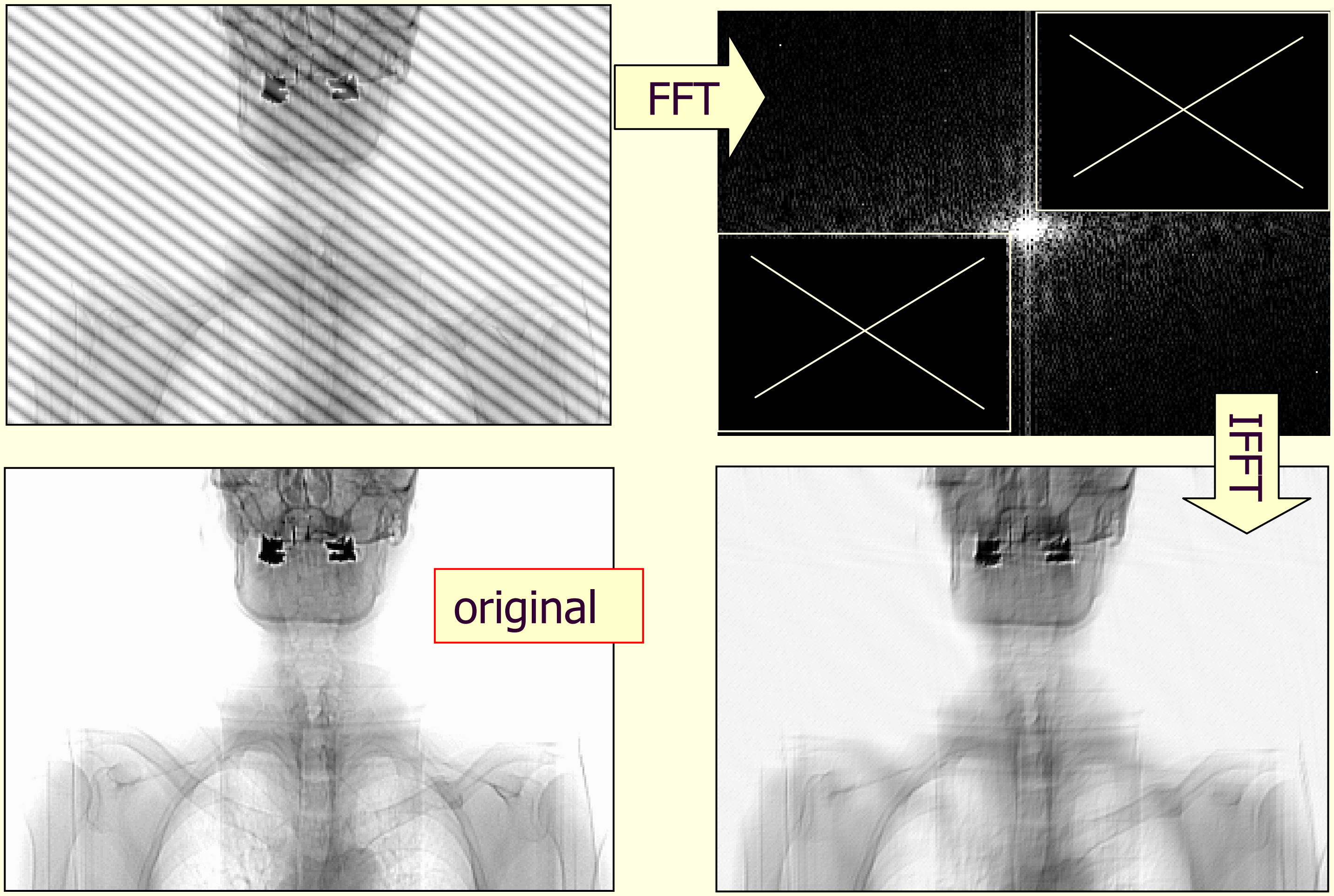
Removing periodic distortions



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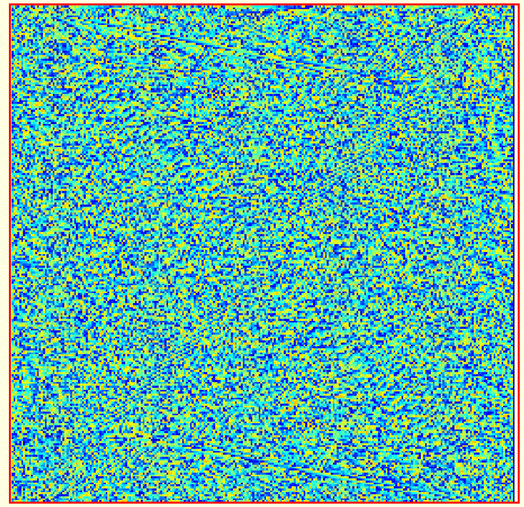
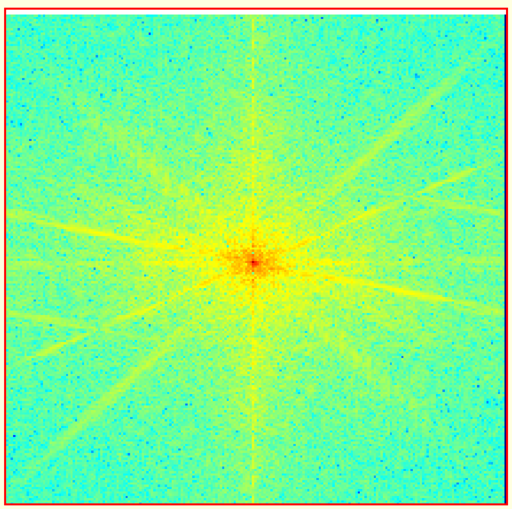
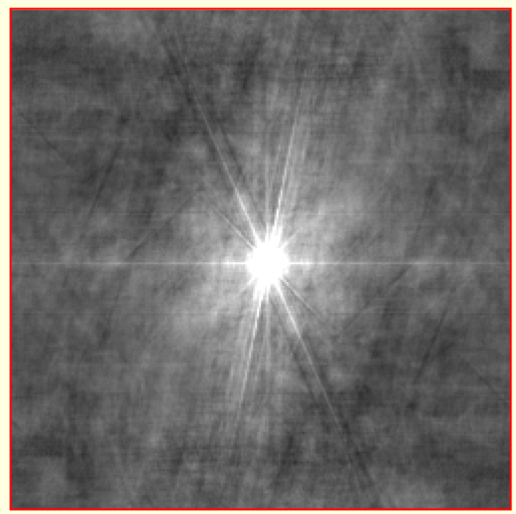
Fourier phase spectrun of an image



$arg\left[F\left(u,v\right)\right]$



$\left|F\left(u,v\right)\right|$



$\Im^{-1}\left\{\left|F\left(u,v\right)\right|\right\}$

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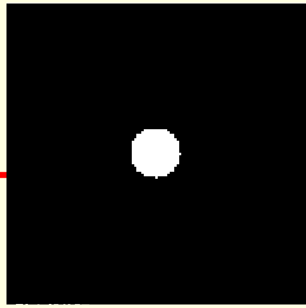
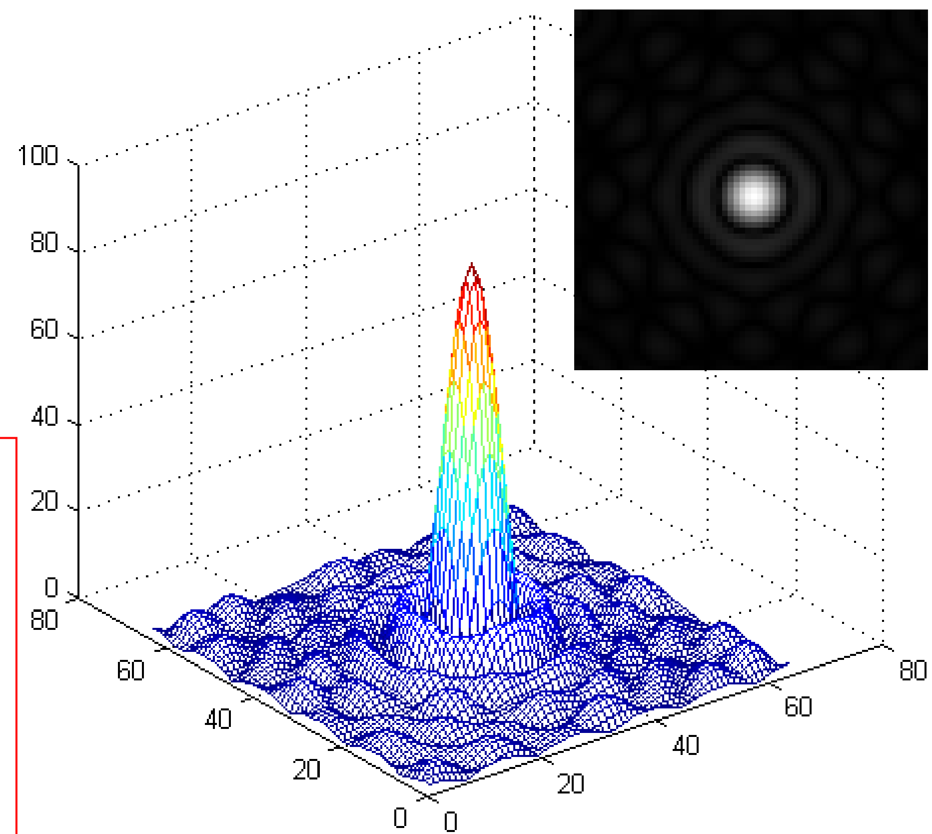
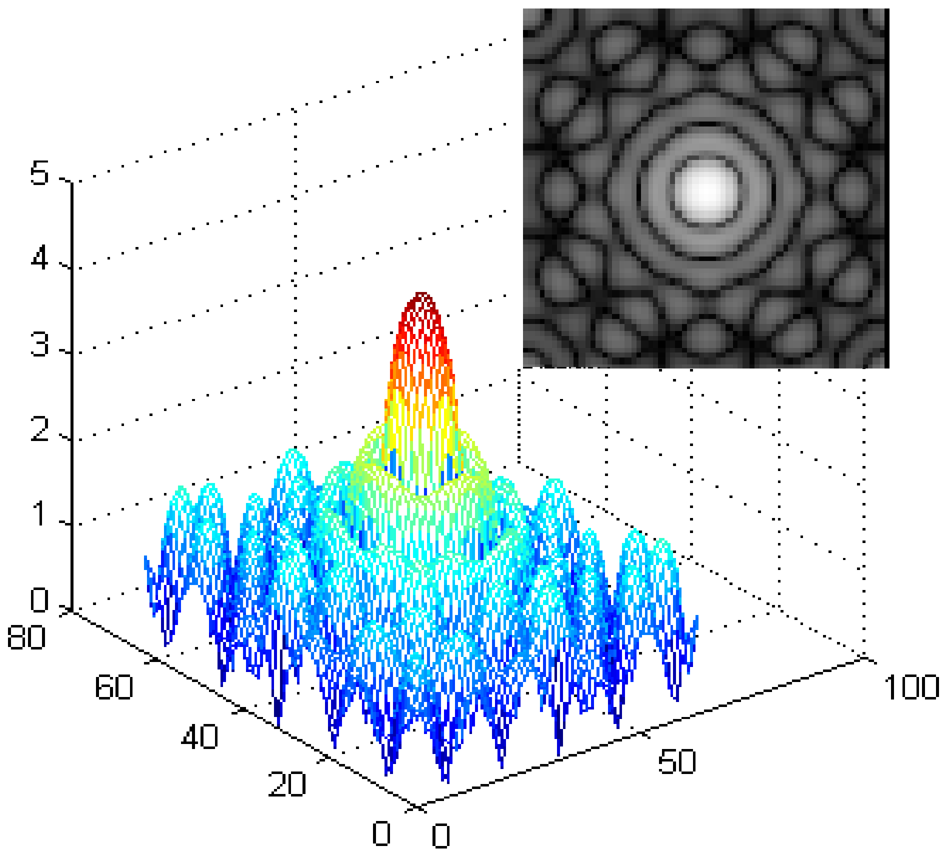


$f(x,y)$

(64x64)





$|F(u,v)|$

$log(1+|F(u,v)|)$

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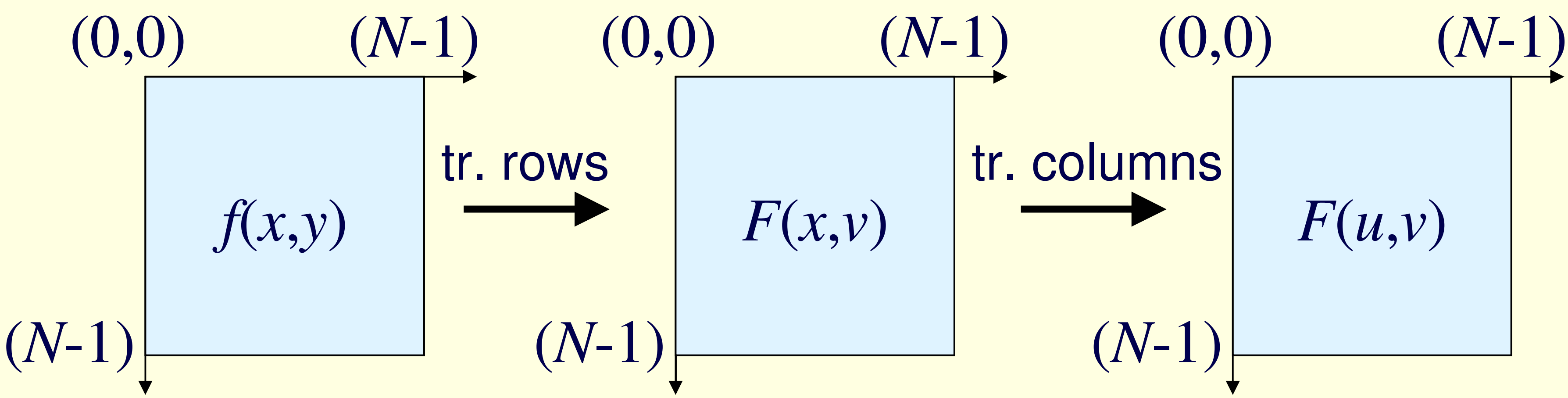
Properties of the two-dimensional Fourier transform

Separability:



Computation of the 2-D Fourier transform as a series of 1-D transforms

Separability of the 2-D Fourier transform

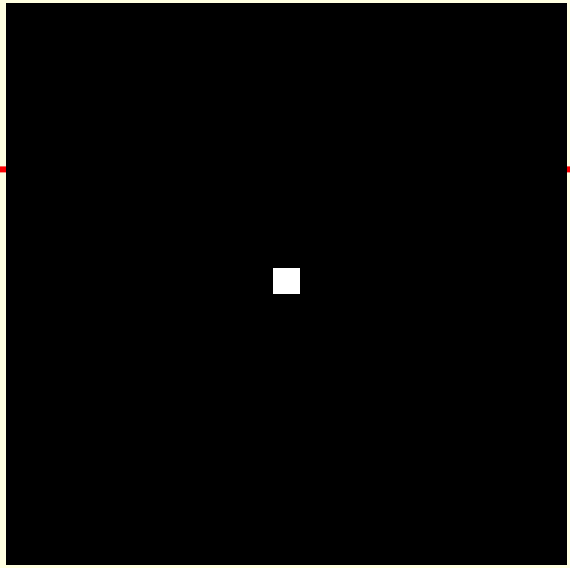
$$F(u,v)=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux+vy)/N}$$

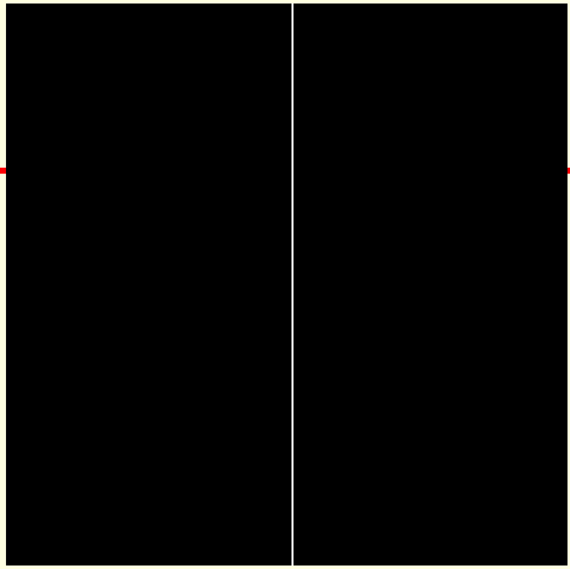
$$F(u,v)=\frac{1}{N^2}\sum_{x=0}^{N-1}e^{-j2\pi ux/N}\overbrace{\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi vy/N}}^{F(x,v)}$$

$$F(u,v)=\frac{1}{N^2}\sum_{x=0}^{N-1}F(x,v)e^{-j2\pi ux/N}$$

Example

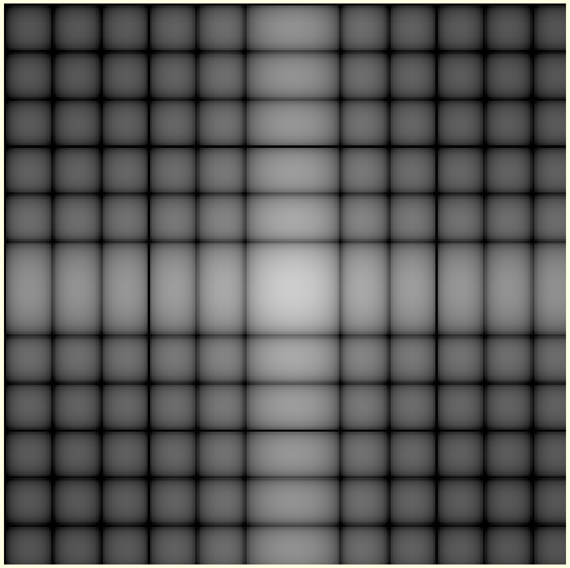
Images

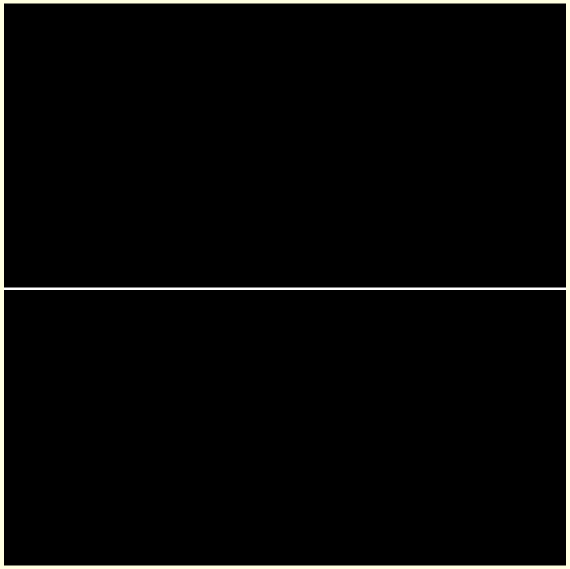




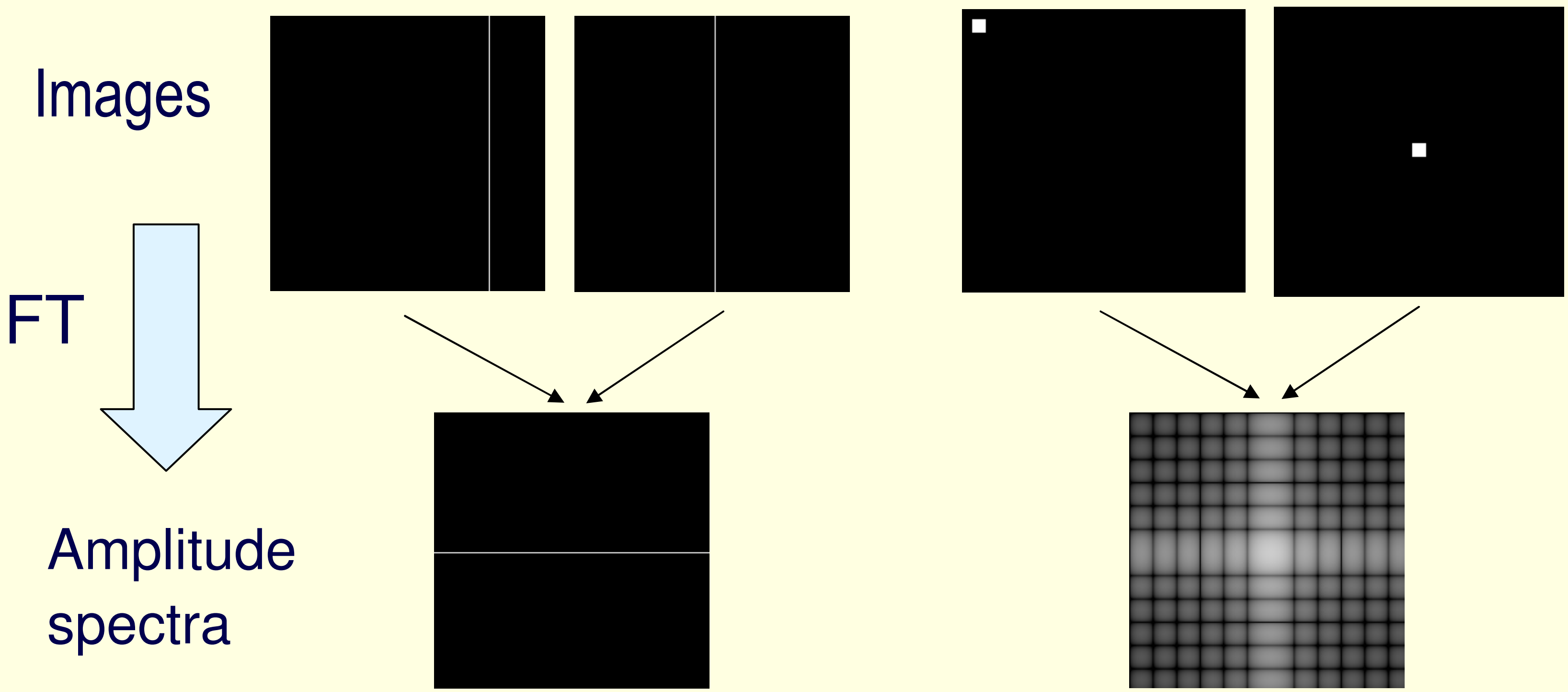
FT

Amplitude spectra





Shift in the spatial domain



$$f\left(x-x_0, y-y_0\right) \Leftrightarrow F\left(u,v\right) exp\left[-\frac{j2\pi\left(ux_0+vy_0\right)}{N}\right]$$

Properties of the two-dimensional Fourier transform

Convolution:

$$\mathcal{F}\{f(x,y)\, g(x,y)\} = F(u,v) * G(u,v)$$

$$\mathcal{F}\{f(x,y) * g(x,y)\} = F(u,v)\, G(u,v)$$

This property is useful in designing digital image filters.

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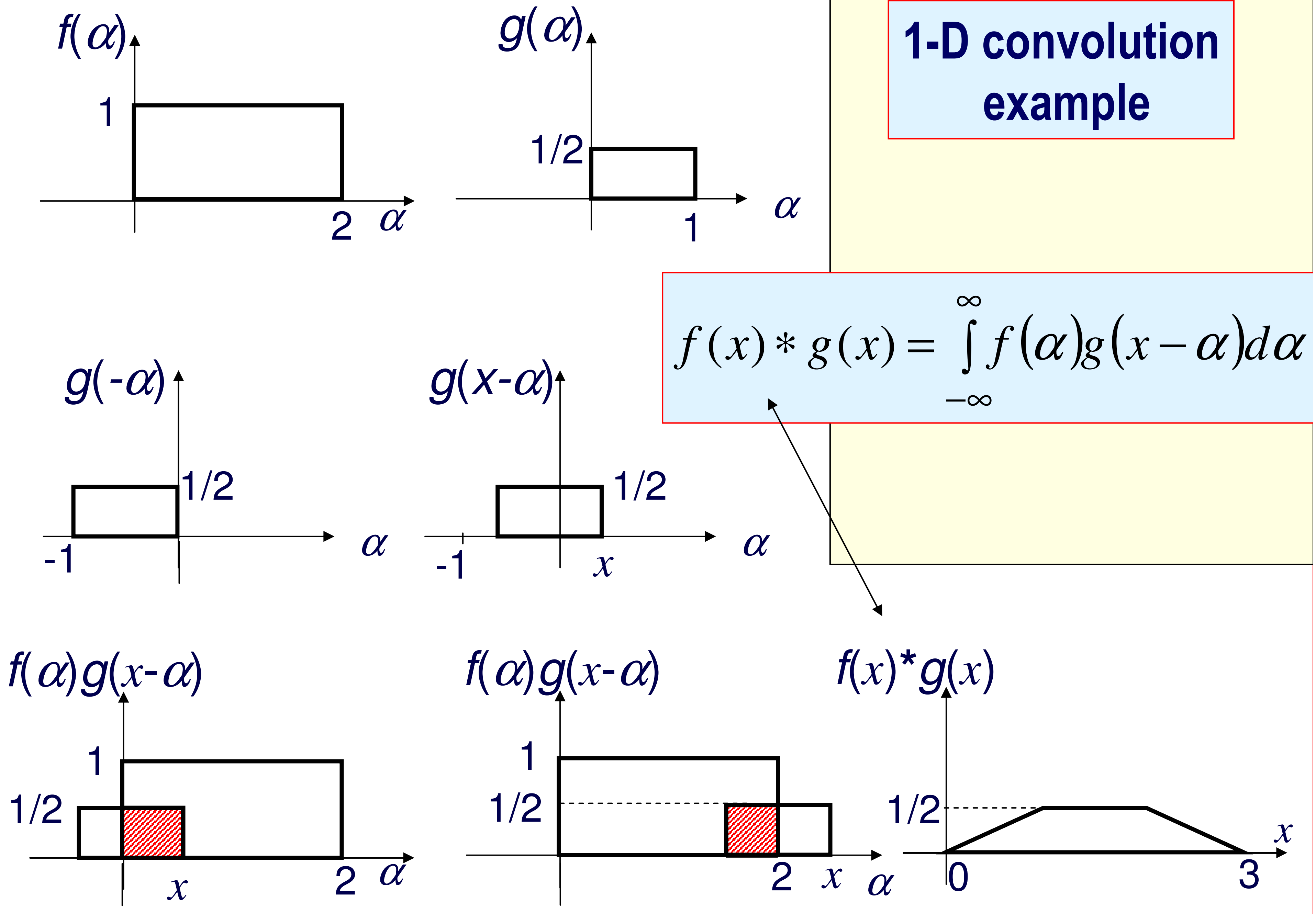
1-D convolution example

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$



2-D convolution of discrete functions

$f(i,j)$, $g(i,j)$ – dicretete 2-D functions of period NxN

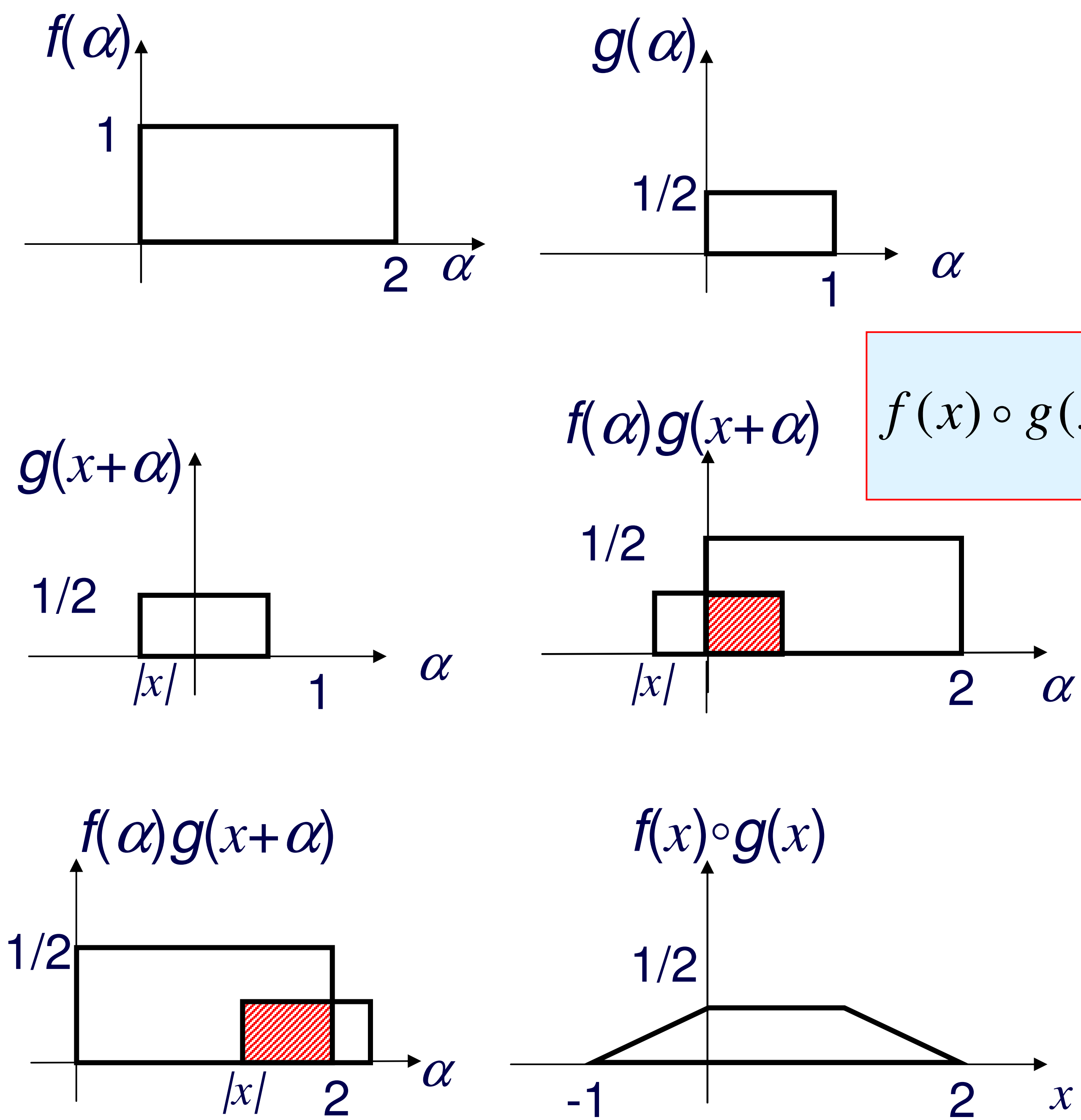
increase periods of $f(i,j)$ and $g(i,j)$ up to $M=2N-1$:

$$f_e(i,k)=\begin{cases} f(i,k) & 0\le i,k\le N-1 \\ 0 & N\le i,k\le M-1 \end{cases} \qquad g_e(i,k)=\begin{cases} g(i,k) & 0\le i,k\le N-1 \\ 0 & N\le i,k\le M-1 \end{cases}$$

$$f_e(i,k)*g_e(i,k)=\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} f_e(m,n)g_e(i-m,k-n)$$

1-D correlation - example

$$f(x) \circ g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x+\alpha) d\alpha$$



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Correlation of 2-D discrete functions

$f(i,j)$, $g(i,j)$ – dicretete 2-D functions of period NxN

Increase the periods as for convolution:

$$f_e(i,k) \circ g_e(i,k) = \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} f_e(m,n) g_e(i+m, k+n)$$

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Periodicity of the FT

$$F(u,v)=F(u+N,v)=F(u,v+N)=F(u+N,v+N)$$

If f(x,y) is a real valued function then:

$$F(u,v) = F^{*}(-u,-v)$$

and:

$$|F(u,v)| = |F(-u,-v)|$$

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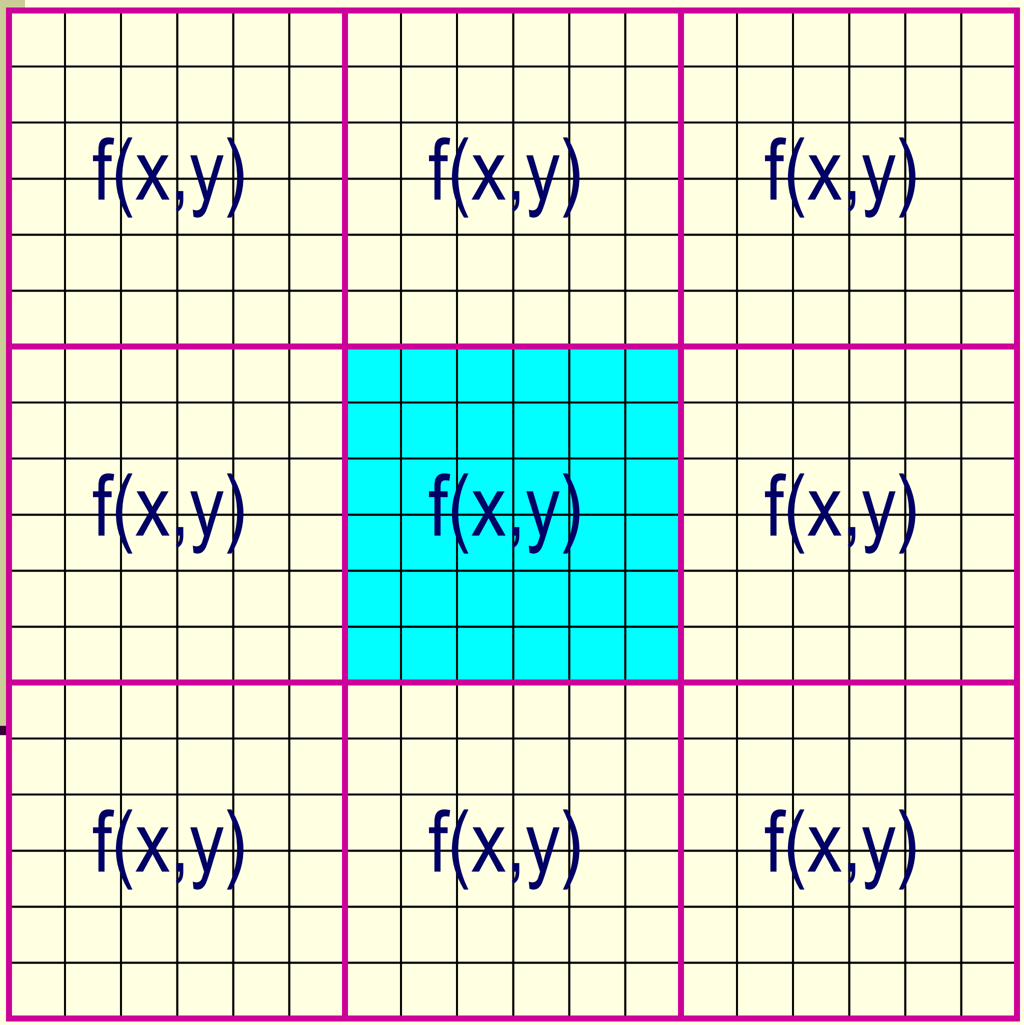
Fourier transform of images



It is assumed the transformed image is a periodic function of period (N, N)

Translation in the spectral domain

$$f(x, y)\exp\left[\frac{j2\pi(u_0 x + v_0 y)}{N}\right] \Leftrightarrow F(u - u_0, v - v_0)$$

This Fourier property is known as the modulation theorem.

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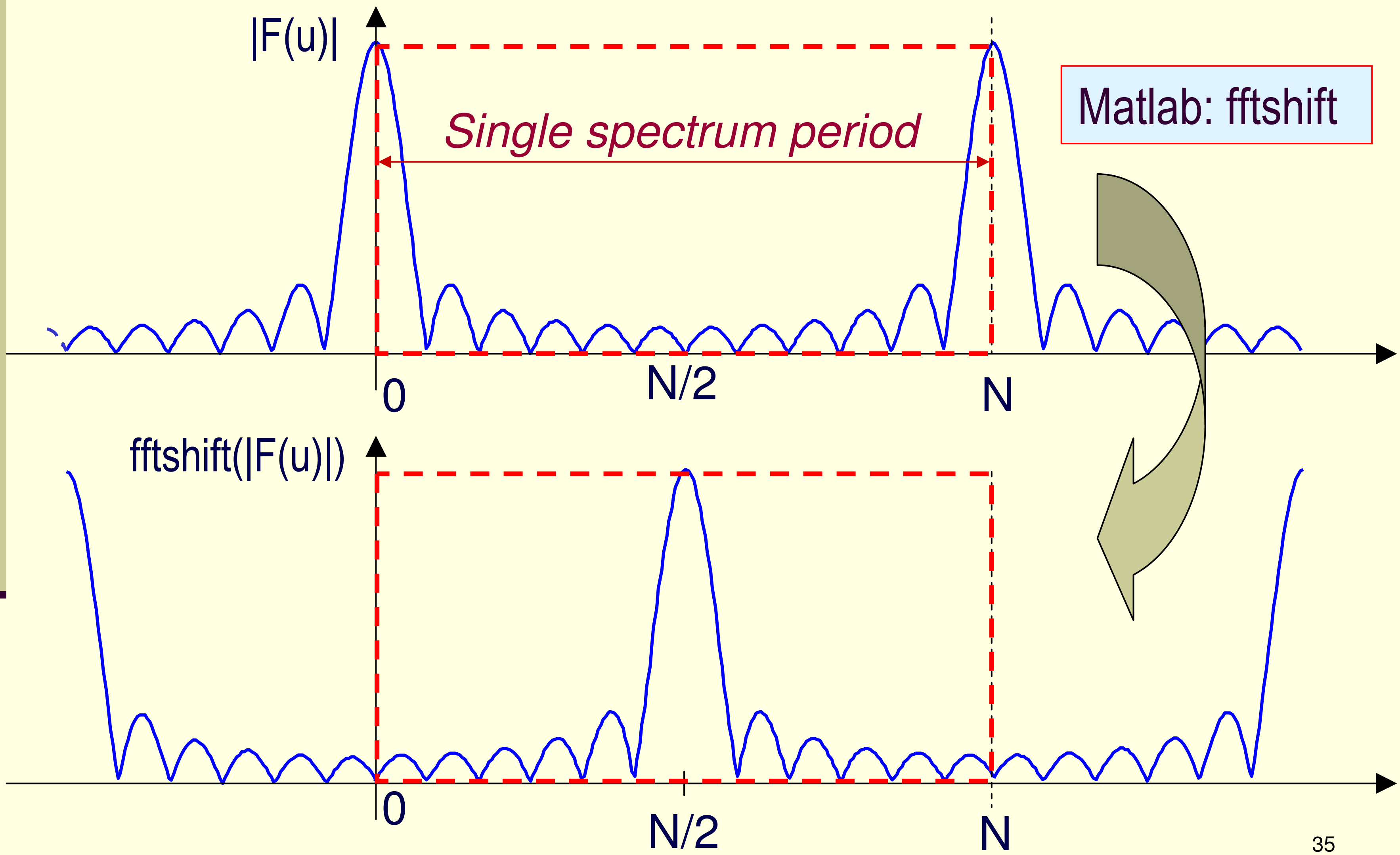
Translation in the spectral domain



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Translation in the spectral domain

$$f(x,y)\exp\left[\frac{j2\pi(u_0x+v_0y)}{N}\right] \Leftrightarrow F(u-u_0, v-v_0)$$

$$for\ u_0 = v_0 = \frac{N}{2} \quad \Leftrightarrow F\left(u-\frac{N}{2}, v-\frac{N}{2}\right)$$

$$f(x,y)\exp\left[\frac{j2\pi(u_0x+v_0y)}{N}\right] =$$

$$= f(x,y)\exp[j\pi(x+y)] = f(x,y)(-1)^{x+y}$$

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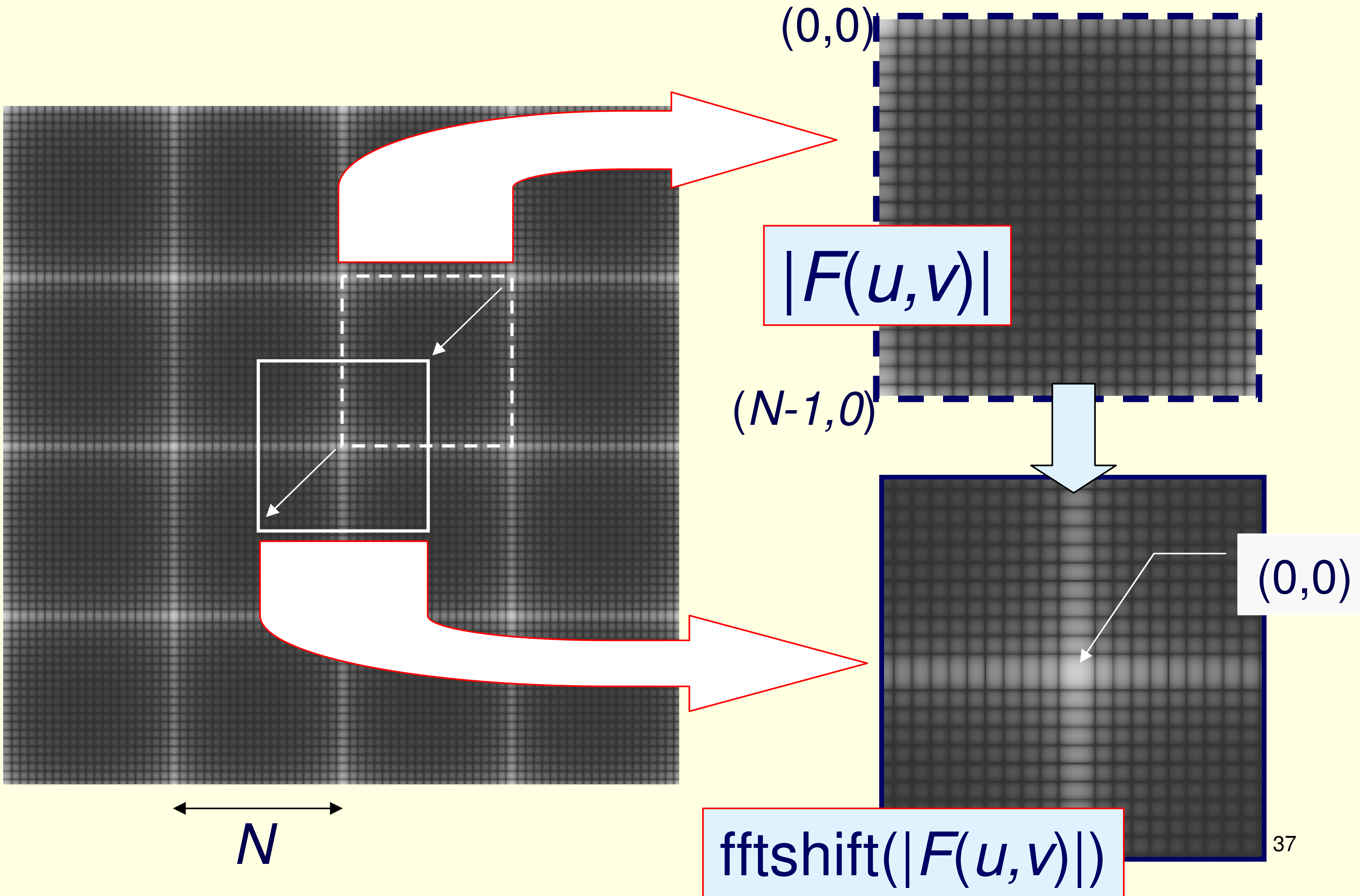
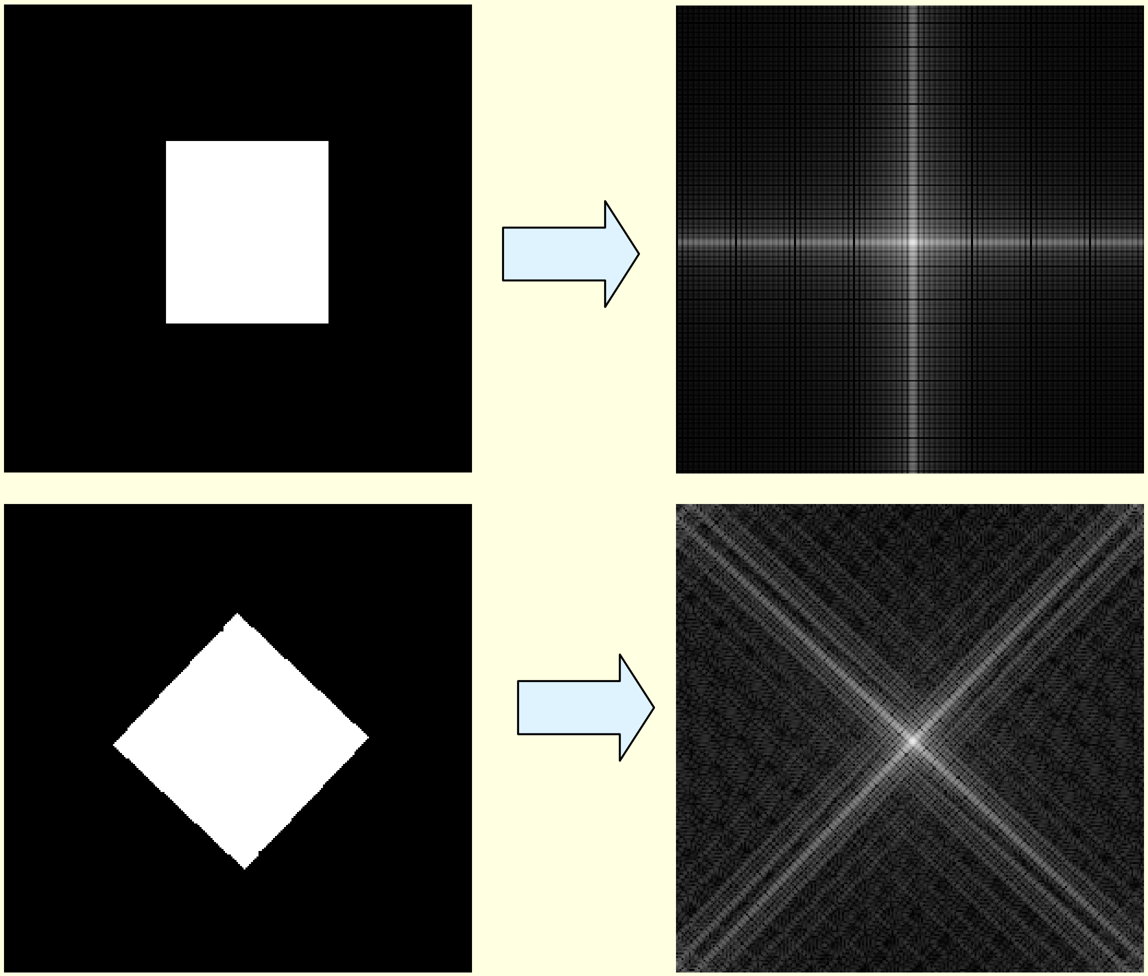
Translation in spectral domain



Rotation



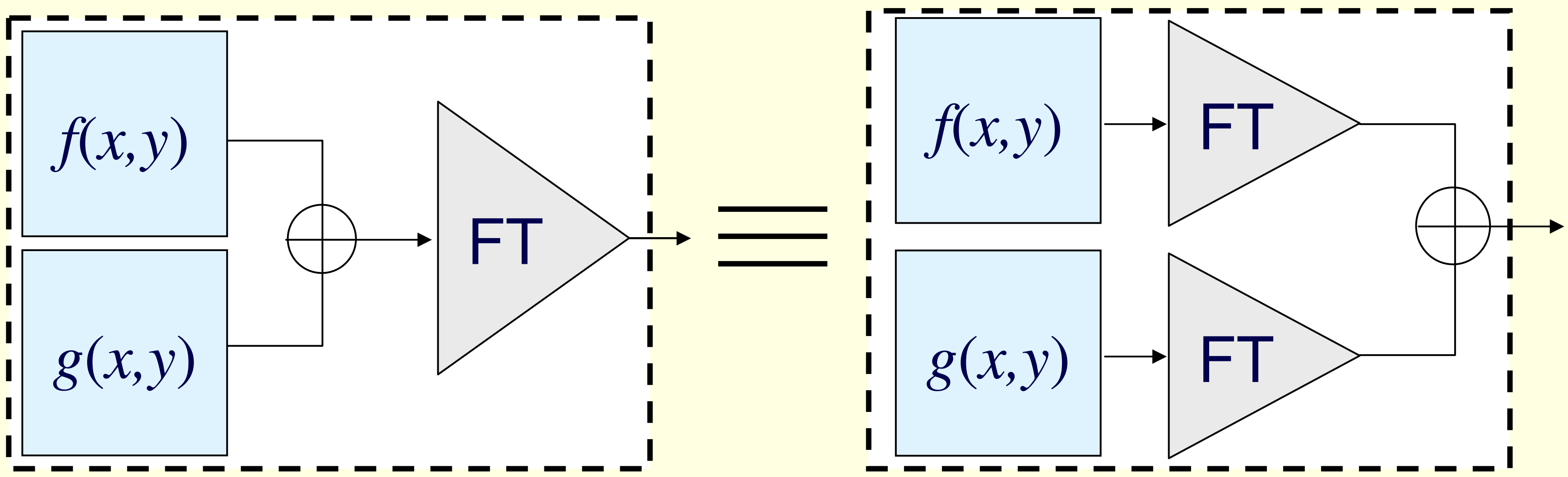
$\theta_0 = 0°$

$\theta_0 = 45°$

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \phi + \theta_0)$$

Linearity

$$\Im\{a\, f(x,y) + b\, g(x,y)\} = a\, F(u,v) + b\, G(u,v)$$



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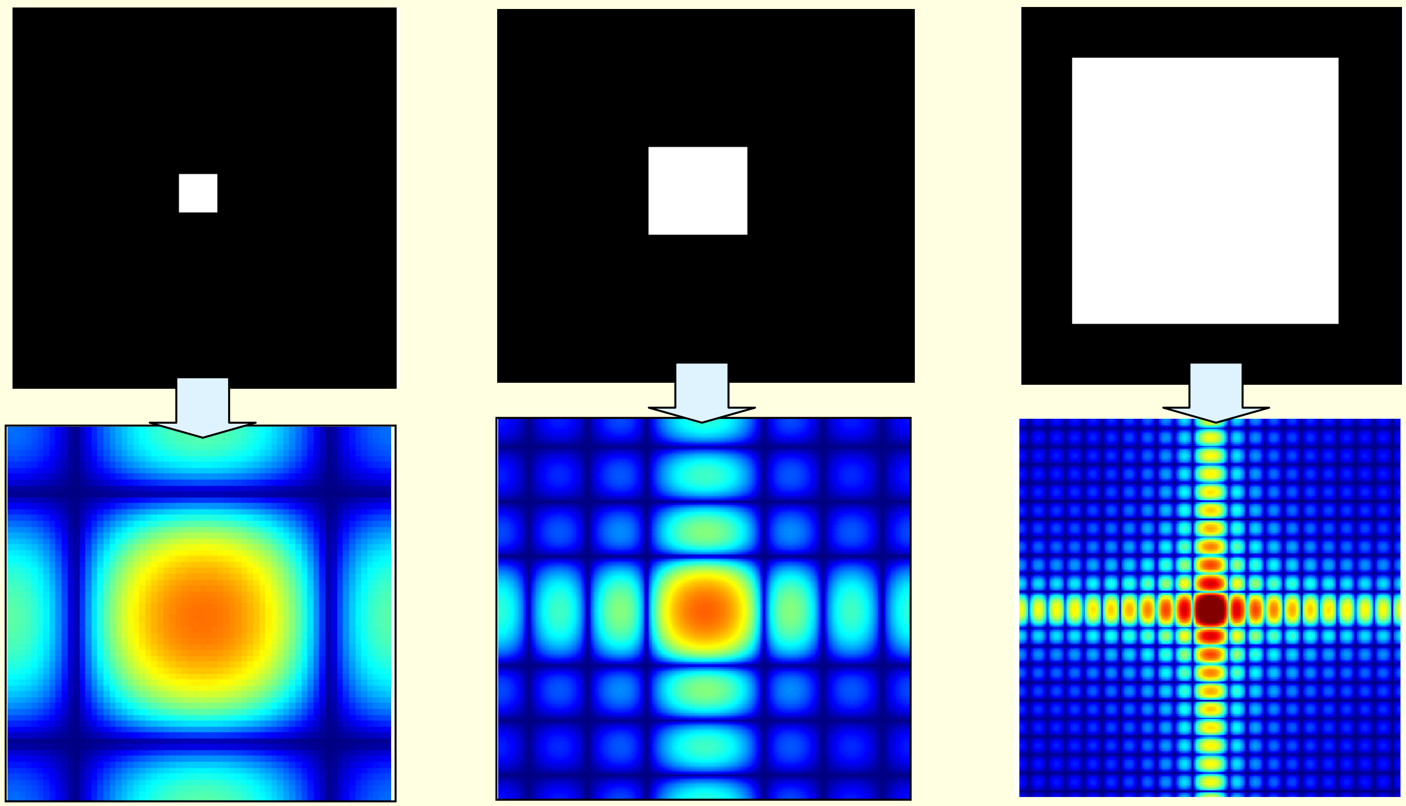
Scaling

$$\Im\{f(ax,by)\} = |ab|^{-1} F(u/a,\ v/b) \qquad a,b \in R$$



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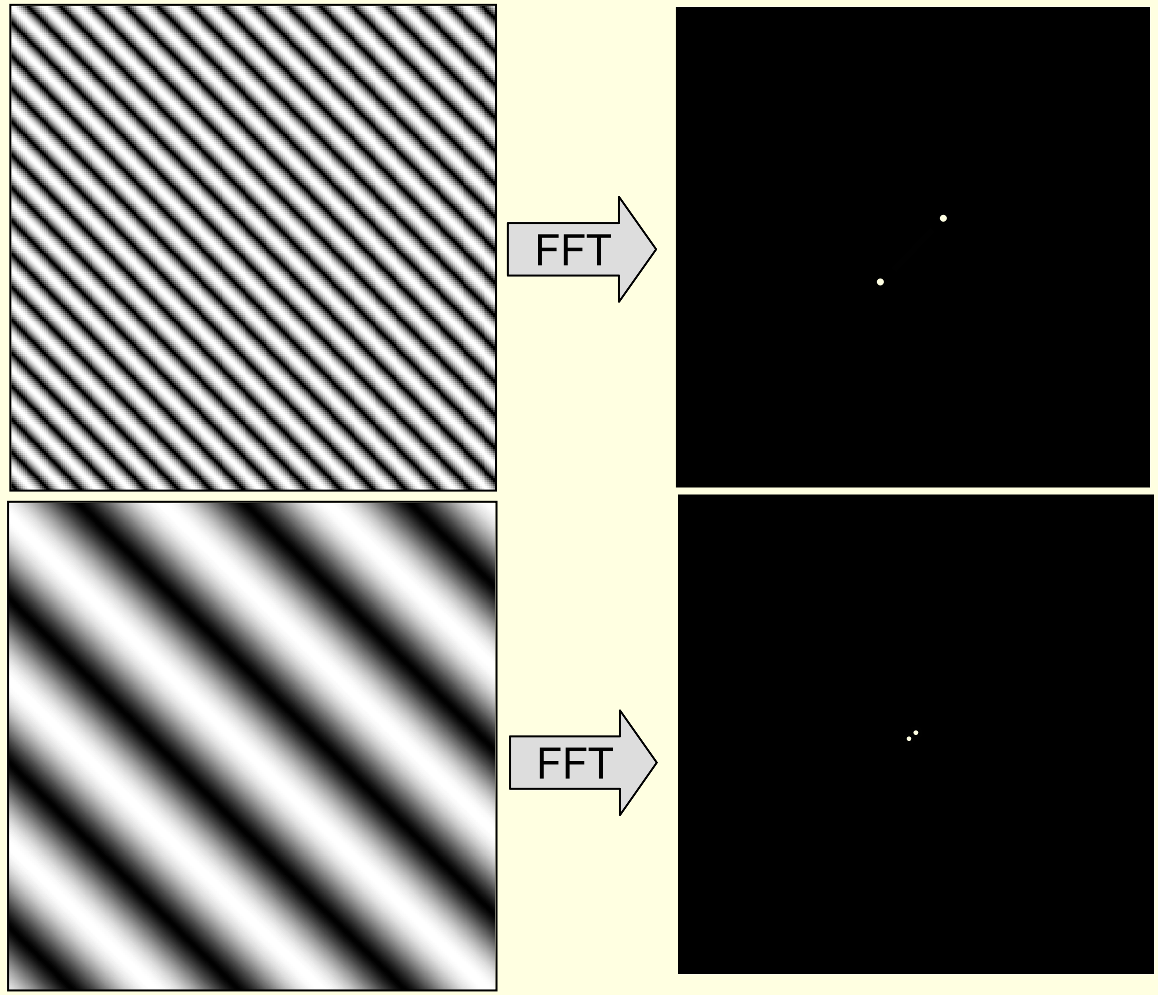
Scaling

$$\Im\{f(ax,by)\} = |ab|^{-1} F(u/a, v/b)$$

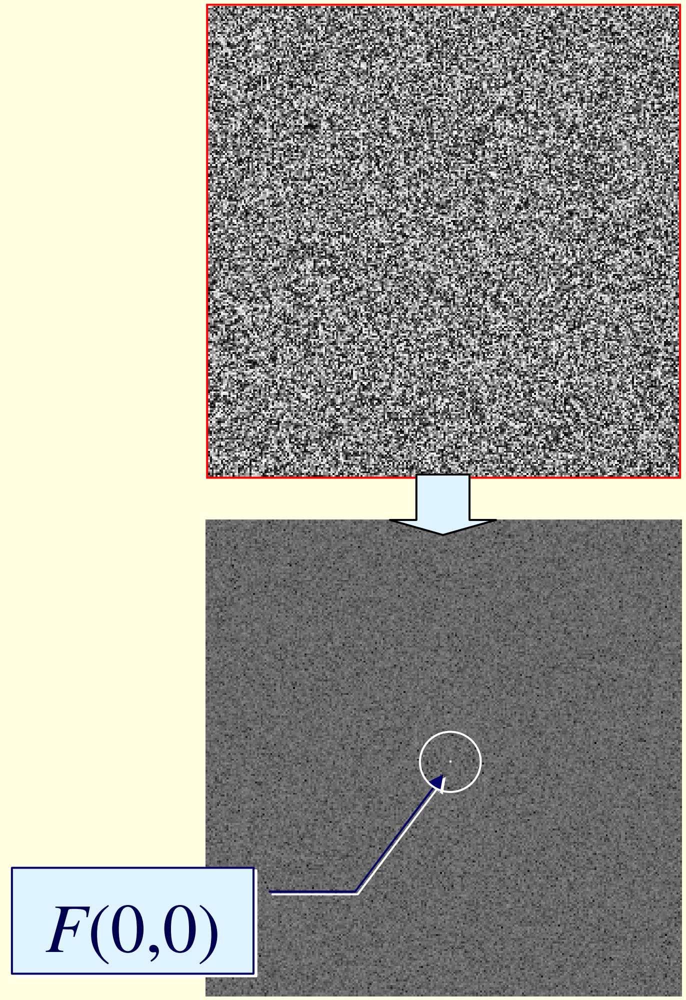


Average value

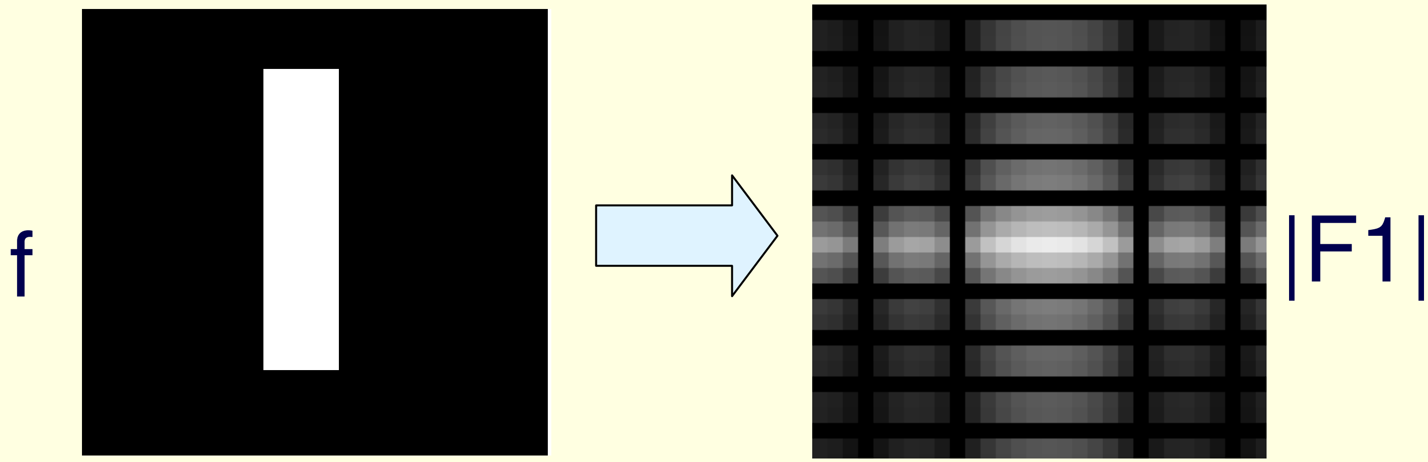
$$\bar{f}(x,y) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)$$

$$F(0,0) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y)$$

$$\bar{f}(x,y) = F(0,0)$$



Fourier transform of an image - examples

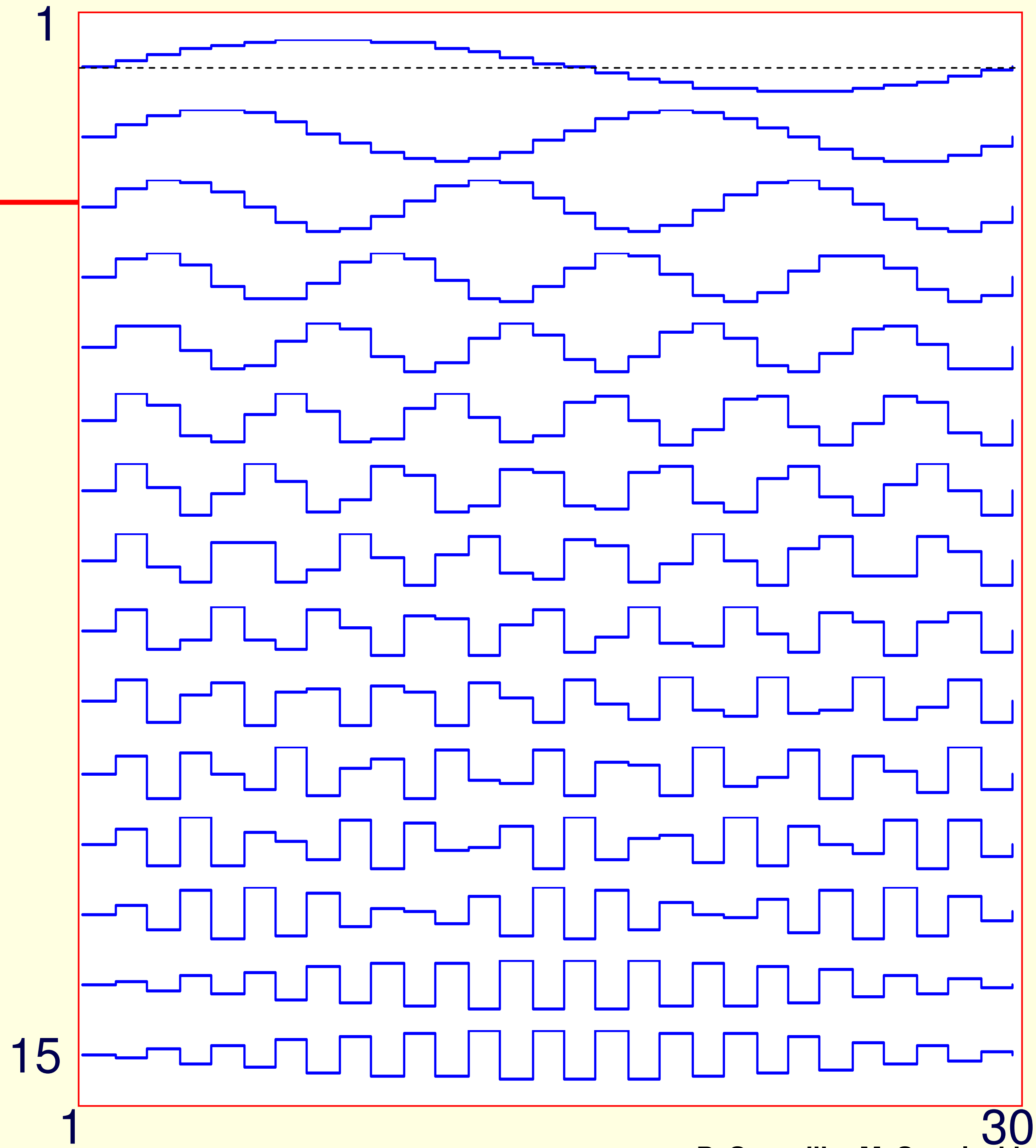



```
f=zeros(30,30);
f(5:24,13:17)=1;

imshow(f,'notruesize')
F=fft2(f);                  %compute 2-D Fourier transform
F1=log(abs(F)+1);           %amplitude spectrum
imshow(F1,[0 5],'notruesize');
```

Discrete Fourier Transform

Basis functions for 30-point Fourier transform (sine component)



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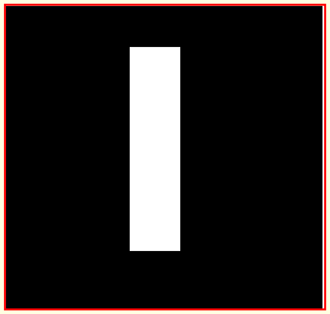
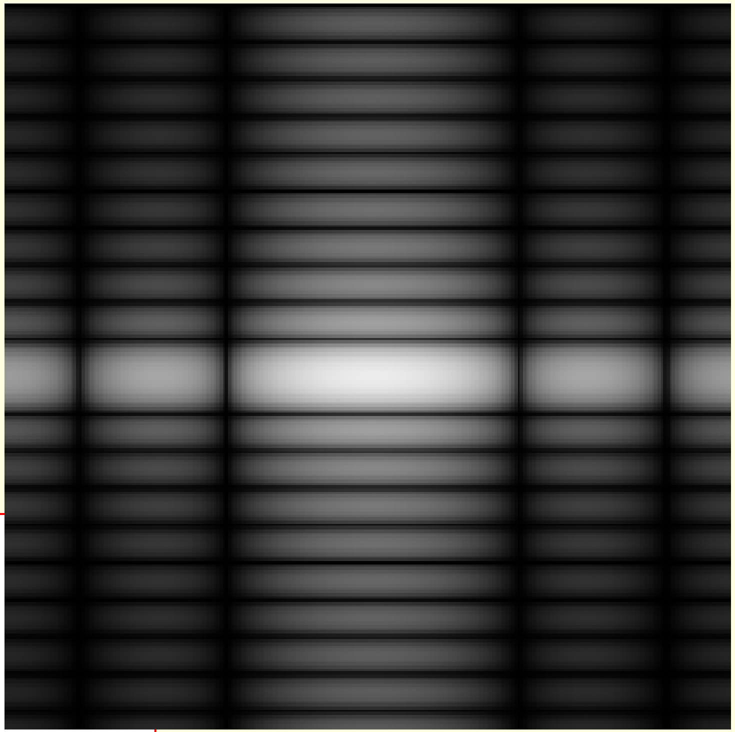
Fourier transform of an image - examples



f



|F2|

```
%better resolution
f=zeros(30,30);
f(5:24,13:17)=1;
F=fft2(f,256,256);
F2=log(abs(F)+1);
imshow(fftshift(F2),[0 5],'notruesize');
```

Fast Fourier Transform, FFT (*succesive doubling method*)

If $N=2^n$, then $N=2^*M$ and one can show that:

$$W_M = e^{-j2\pi/M}$$

$$F_{even}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x)W_M^{ux}, \qquad F_{odd}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)W_M^{ux}$$

$$F(u) = \frac{1}{2}[F_{even}(u) + F_{odd}(u)W_{2M}^{u}], \qquad u = 0,1,\ldots,M-1$$

$$F(u+M) = \frac{1}{2}[F_{even}(u) - F_{odd}(u)W_{2M}^{u}], \qquad u = 0,1,\ldots,M-1$$

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Fast Fourier Transform, FFT

FFT is an efficient algorithm for computing Discrete Fourier Transform

FFT exploits periodicity of complex sinusoidals:

$$e^{j2\pi kn/N} = \left(e^{j2\pi/N}\right)^{kn} = W^{kn} \quad where:\ W = e^{j2\pi/N}$$

$$for:\ n = 7,\ k = 5,\ N = 32$$

$$W^{(5)(7)} = W^{35} = \left(W^{32}\right)^{3} = W^{3}$$

$$gdy\dot{z}:W^{(k+N)n} = W^{k(n+N)} = W^{kn}$$

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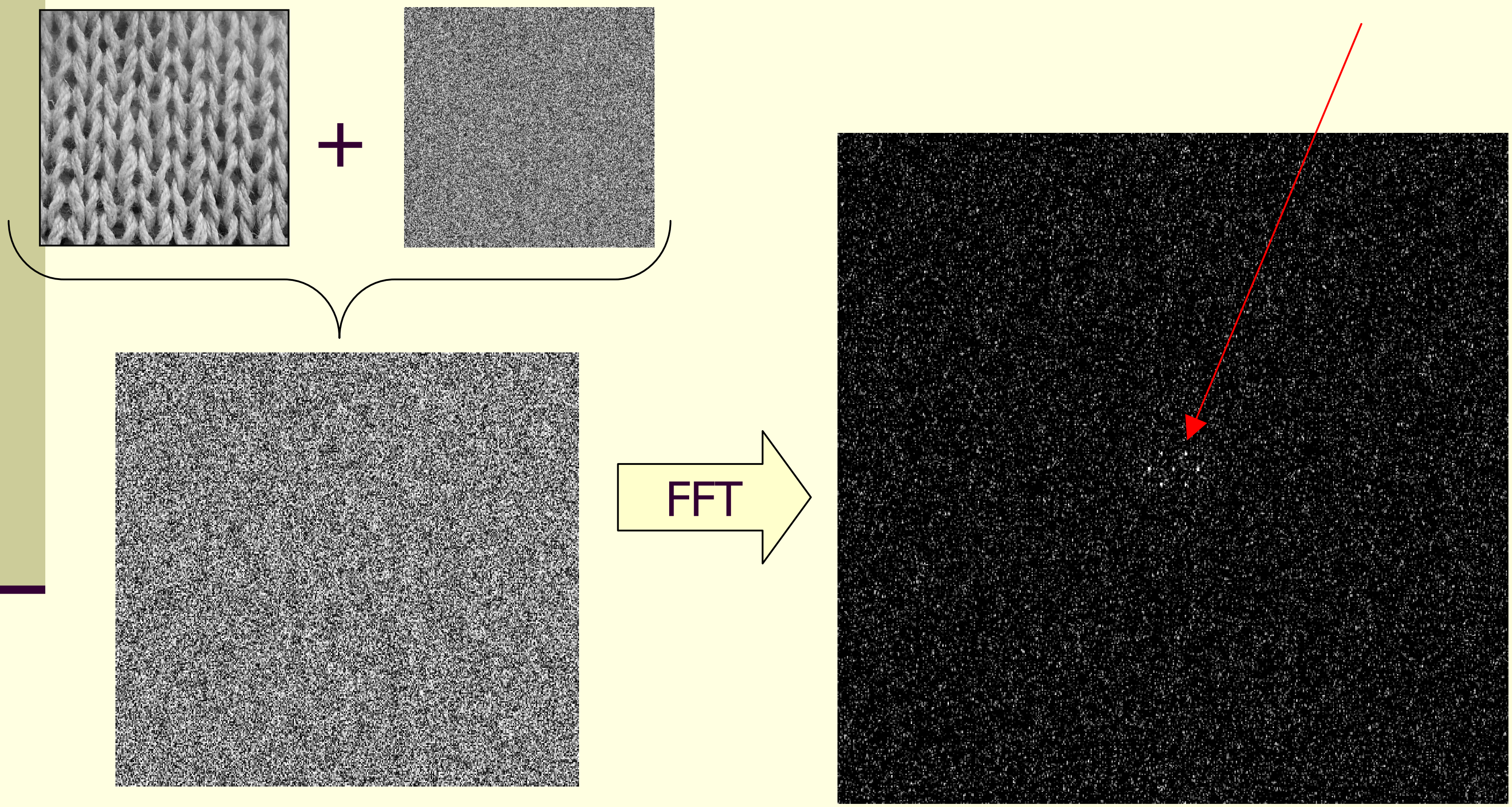
Comparison of TF and FFT

N	N^2 (FT)	NlogN (FFT)	Advantage N/logN
16	256	64	4
256	65535	2048	32
512	262144	4608	64
2048	~4e6	22528	186

Detecting periodic image content



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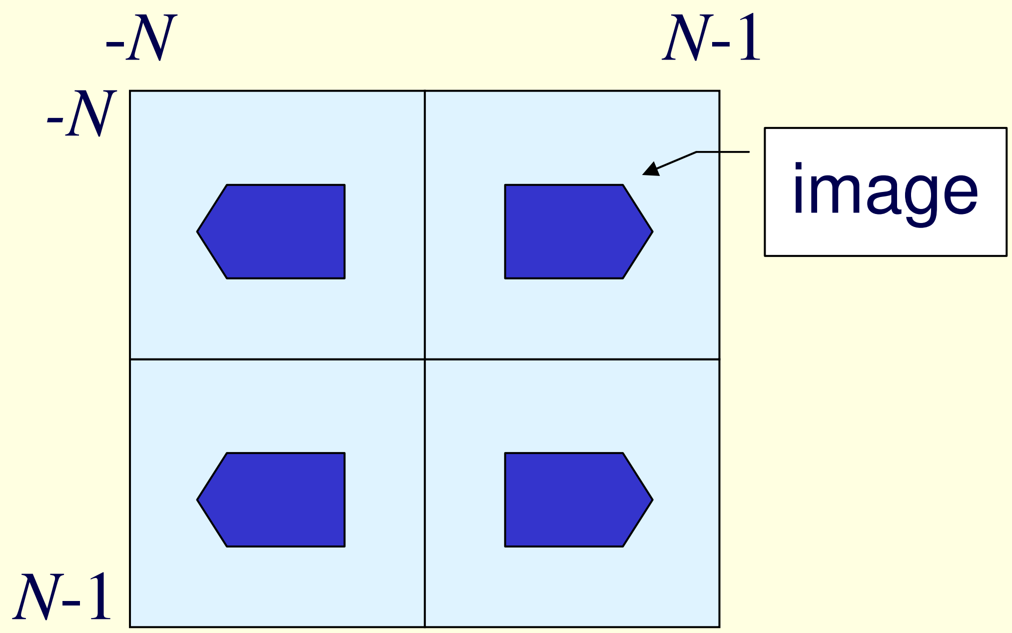
Discrete Cosine Transform (DCT)

$$F(u,v)=\frac{2}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1}f(x,y)cos\left[\frac{\pi u(2x+1)}{2N}\right]cos\left[\frac{\pi v(2y+1)}{2N}\right]$$

for: $u,v=1,2,\ldots,N-1$

Fourier spectrum of a real valued and symmetric function has real valued coeffcients, ie. only those associated with the cosine components of the Fourier series



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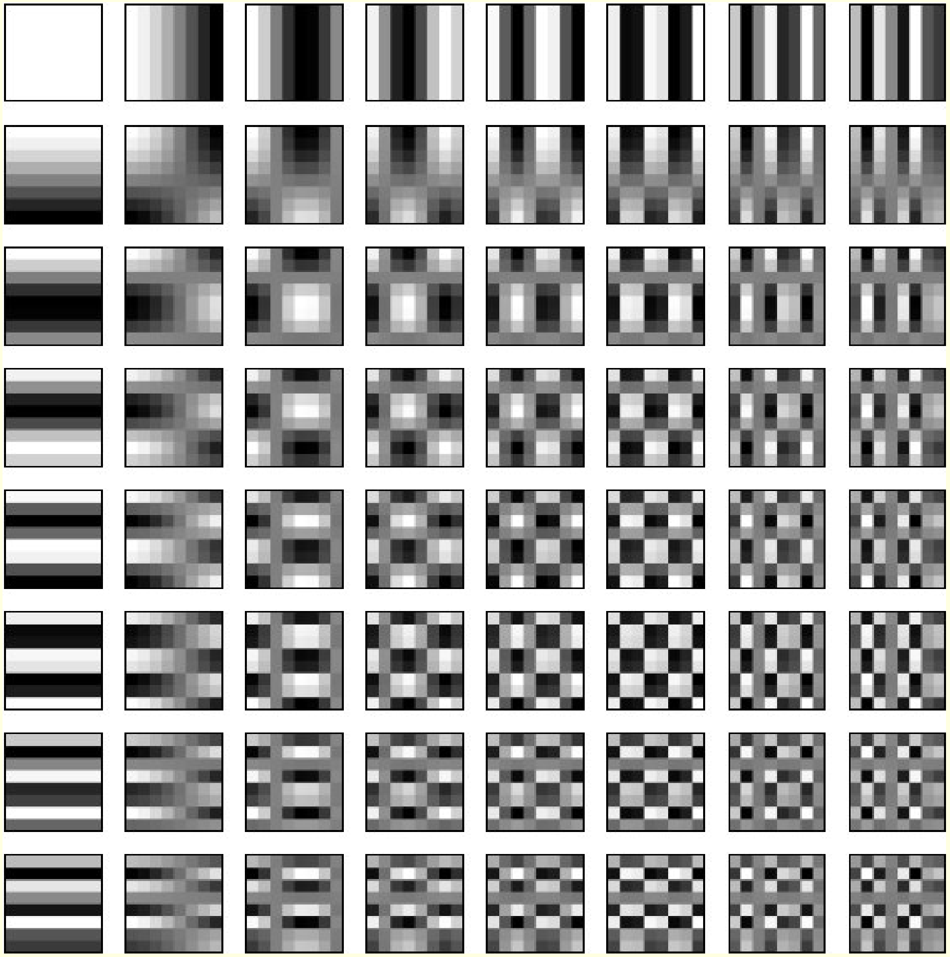
DCT basis functions

DCT basis functions for 8x8 image blocks



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Discrete Cosine Transform (DCT)

'autumn' image

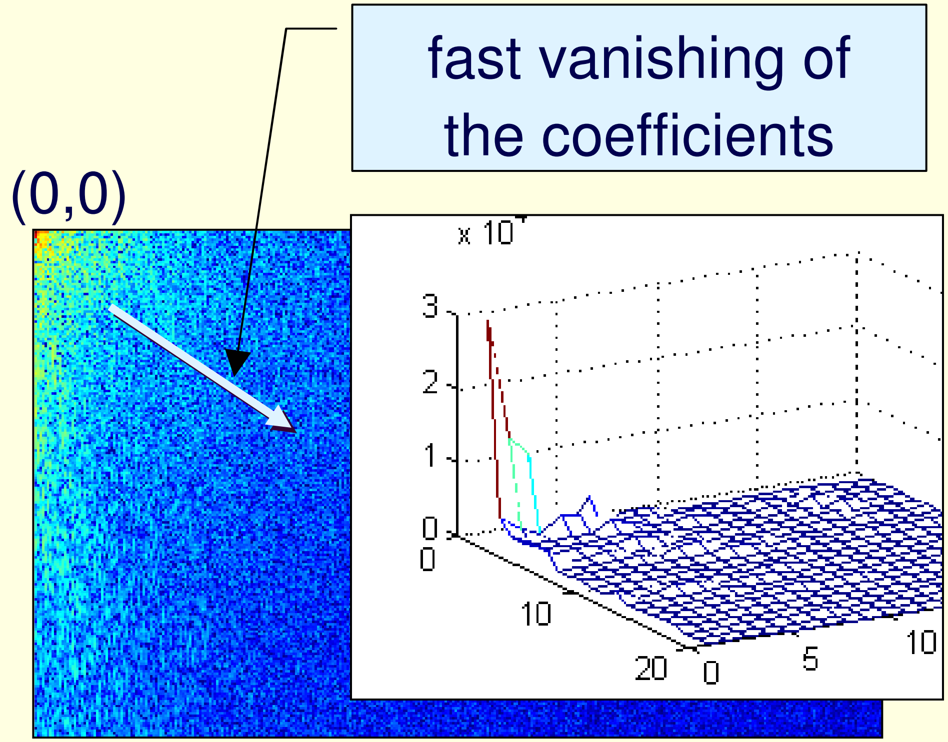

image cosine transform

The JPEG image compression standard is based on DCT

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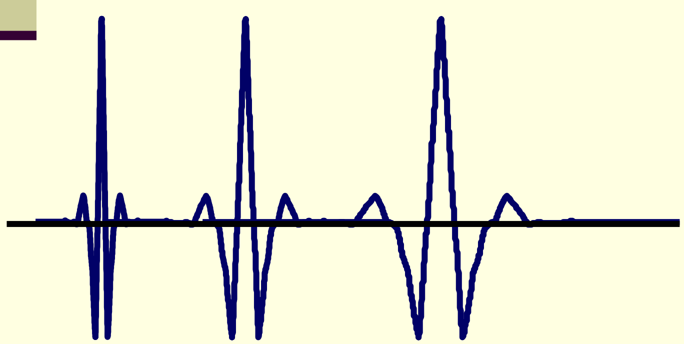
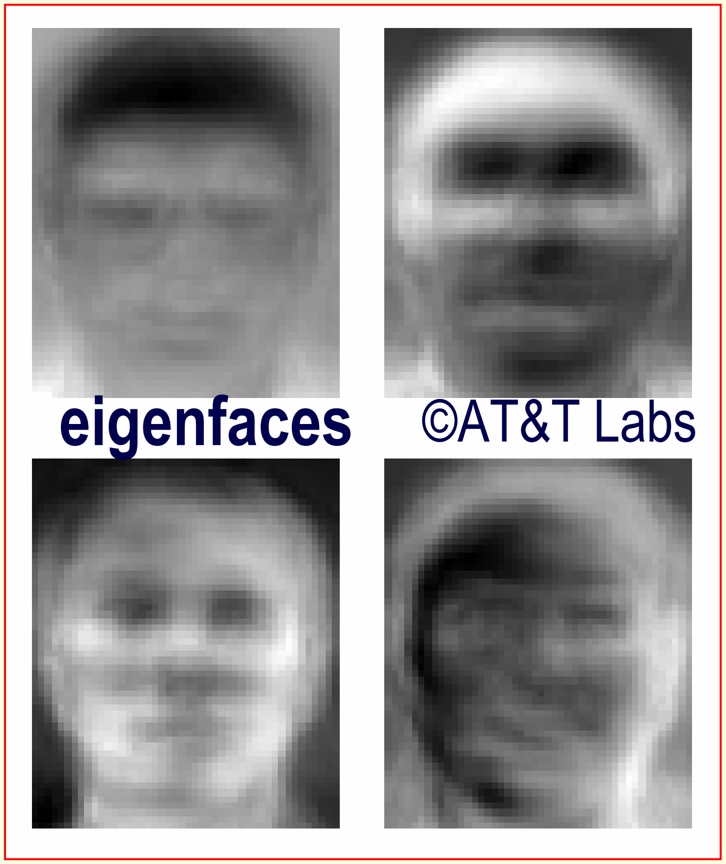
Other image transforms

- the **Karhunen-Loeve** transform - equivalent to the **PCA** (*Principal Component Analysis*)
- the wavelet transform is used in JPEG-2000 image coding standard





Other image transforms